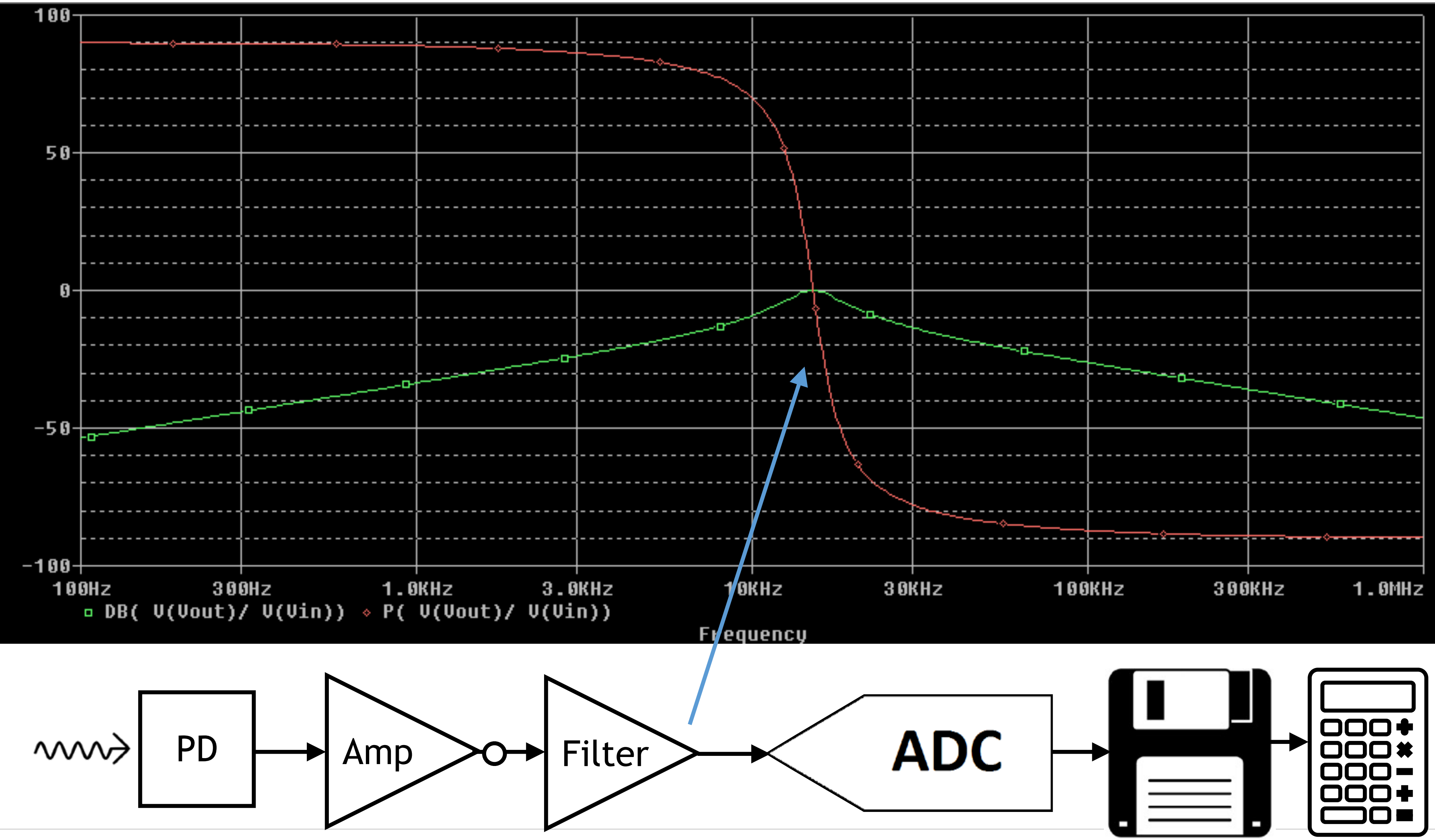
PHYS127AL Lecture 3

David Stuart, UC Santa Barbara

Complex impedance, filters, inductors



Review

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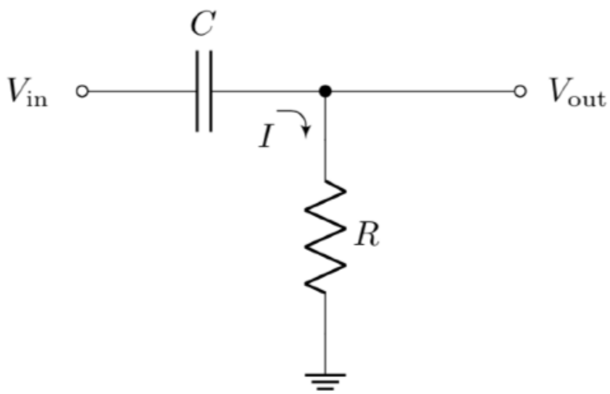
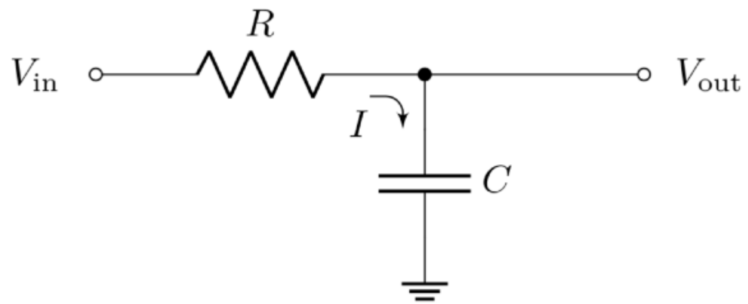
Input and output impedance & rule for chaining circuit stages

Capacitors in time domain: $q=CV \Rightarrow I = CdV/dt$





$I = C\, dV/dt$

$V_{out}/R = C\, d(V_{in}-V_{out})/dt$

If $dV(t)/dt \ll dV_{in}/dt$ then

$V_{out}(t) = RC\, dV_{in}/dt$

Differentiator

$I = C\, dV/dt$

$(V_{in}-V)/R = C\, dV/dt$

$V(t) = V_{in}\,(1-e^{-t/RC})$

$dV = V_{in}(t)\, dt\, /RC$, if $V(t) \ll V_{in}(t)$

$\Rightarrow V(t) = \int V_{in}(t)\, dt\, /RC$

Integrator

Outline

Generalize Ohm's law with complex impedance

Frequency domain description

Filters

Inductors

Phase diagrams

Ohm's law for capacitors

We can find an IV relationship for a capacitor. Let's assume that V(t) is sinusoidal; we can then form any AC signal from a Fourier sum of these.

I'll actually use a cosine for reasons that will become clear later.

Suppose the voltage across the capacitor is

$$V(t) = V_0 \cos \omega t$$

where $\omega=2\pi f$. Then the current through it is

$$I(t) = C\, dV(t)/dt = -\omega C V_0 \sin \omega t$$

So we have a relation between the magnitudes of V and I

$$|V| = |I|\,(1/\omega C)$$

This is similar to Ohm's law; we can identify *reactance* of a capacitor as $X_C = 1/\omega C$. But it is not the full story because the *phase* of the current differs from the phase of the voltage.

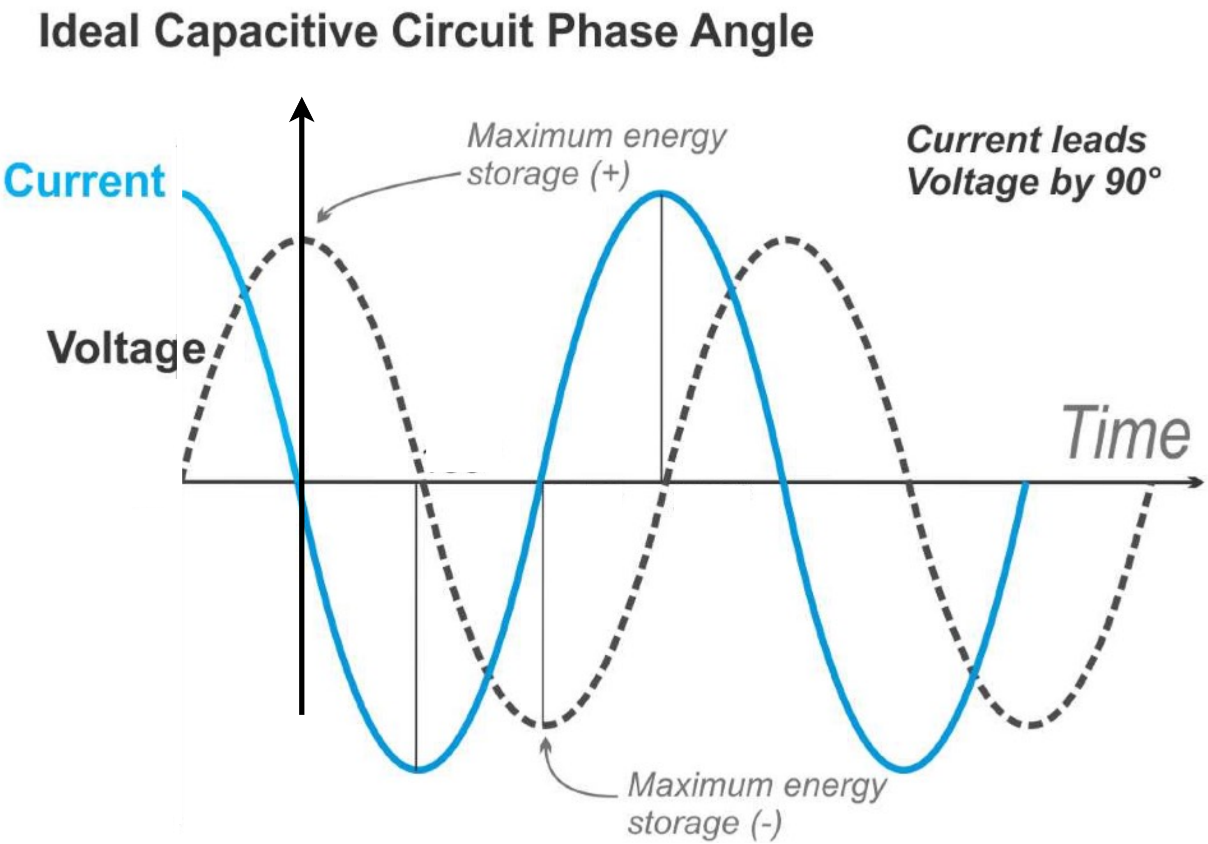
Impedance is the general term for resistance or reactance

Note that Z is sometimes used for impedance instead of X.

Ohm's law for capacitors

$|V| = |I| (1/\omega C)$

This is similar to Ohm's law, and we can identify $1/\omega C$ as being like the "resistance of a capacitor". But it is not the full story because the *phase* of the current differs from the phase of the voltage.



$V(t) = V_0 \cos \omega t$

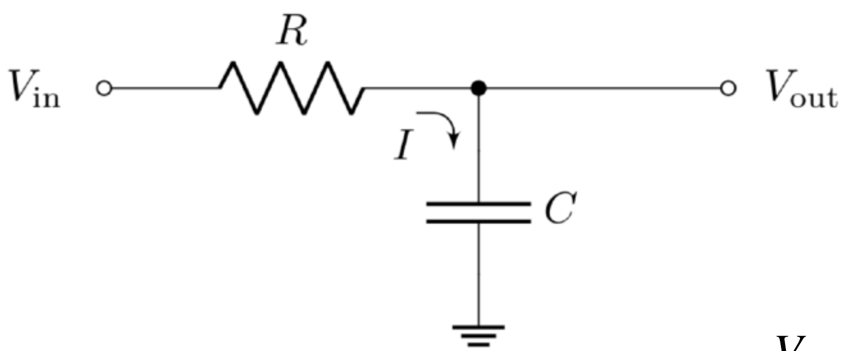
$I(t) = C\, dV(t)/dt$

$= -\omega C V_0 \sin \omega t$

Intuition for impedance of a capacitor = 1/ωC

The 1/ωC means that the impedance of a capacitor depends on the frequency, ω, of the signal.

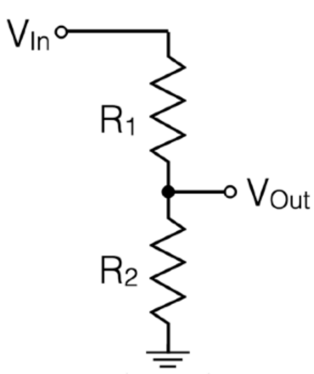
Low impedance for high frequency AC and high impedance for DC.

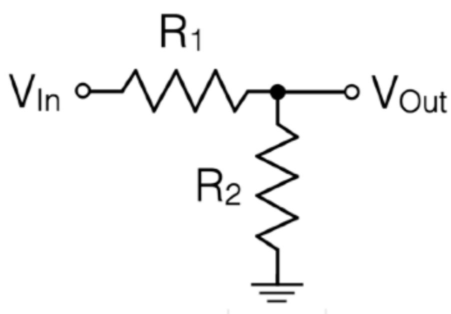


This simple rule helps us understand this circuit intuitively.

It is just a voltage divider.

$$V_{\text{Out}} = V_{\text{In}} \frac{R_2}{R_1 + R_2} = V_{\text{In}} \frac{X_2}{X_1 + X_2} = V_{\text{In}} \frac{1/(\omega C)}{R + 1/(\omega C)}$$





V_{Out}/V_{In} is large (≅1) if X_2 is large compared to X_1 and

V_{Out}/V_{In} is small (≅0) if X_2 is small compared to X_1.

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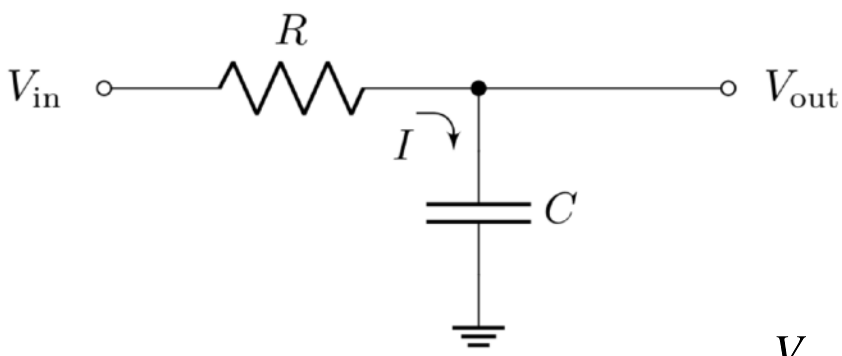

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First consider a DC signal at V_{In}, i.e., $\omega=0$. What is V_{out}?

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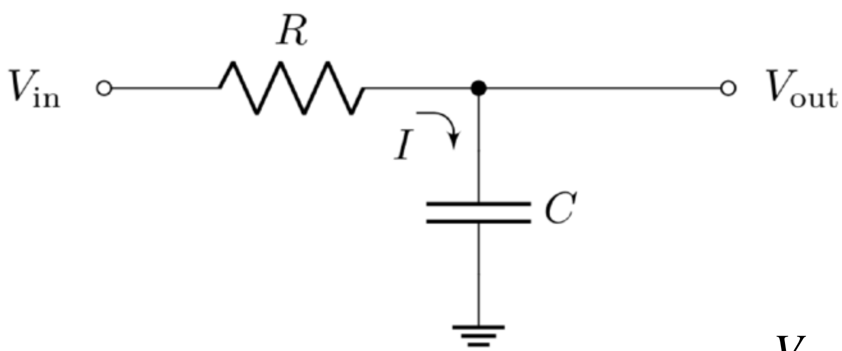

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First consider a <u>DC signal</u> at V_{in}, i.e., $\omega=0$. What is V_{out}?

$X_C \rightarrow \infty$, so $V_{out} = V_{in}$.

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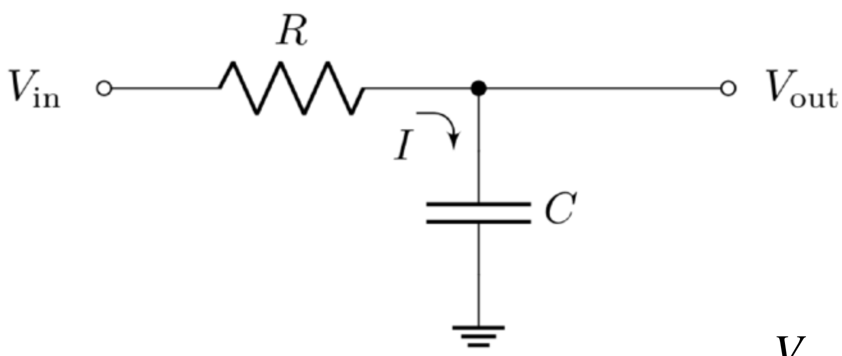

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Next consider a high frequency AC signal at V_{in}, i.e., $\omega \rightarrow \infty$. What is V_{out}?

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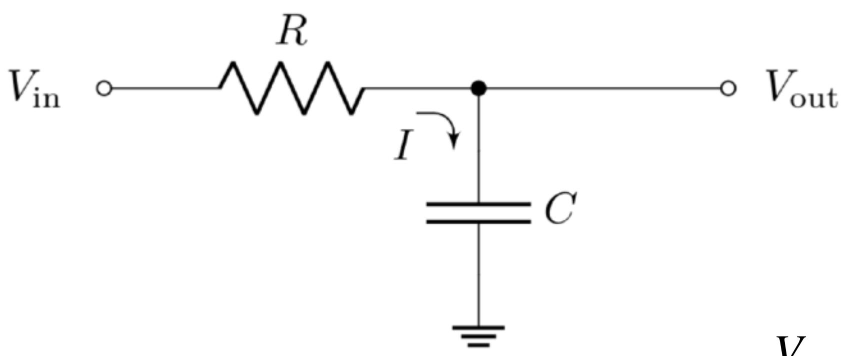

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First consider a <u>DC signal</u> at V_{in}, i.e., ω=0. What is V_{out}?

$X_C = 1/(\omega C) \to \infty$, so $V_{out} = V_{in}$.

Next consider a <u>high frequency AC signal</u> at V_{in}, i.e., $\omega \to \infty$. What is V_{out}?

$X_C = 1/(\omega C) \to 0$, so $V_{out} \to 0$.

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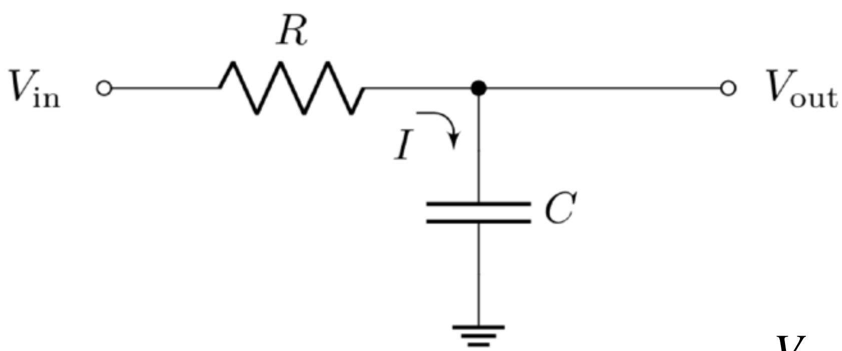

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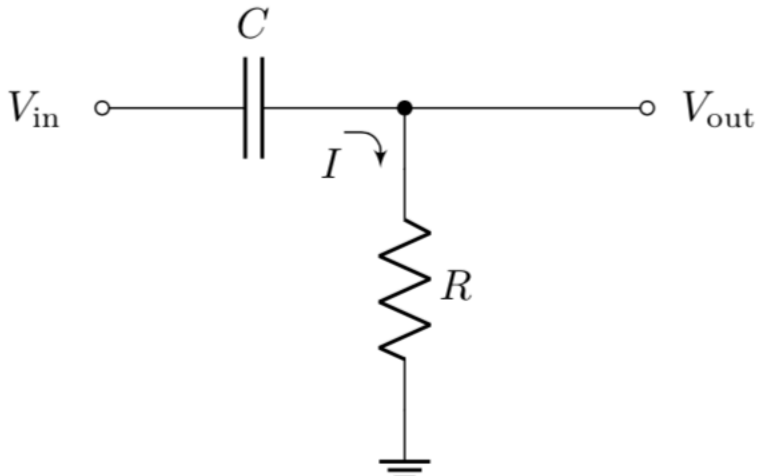

How about this one?

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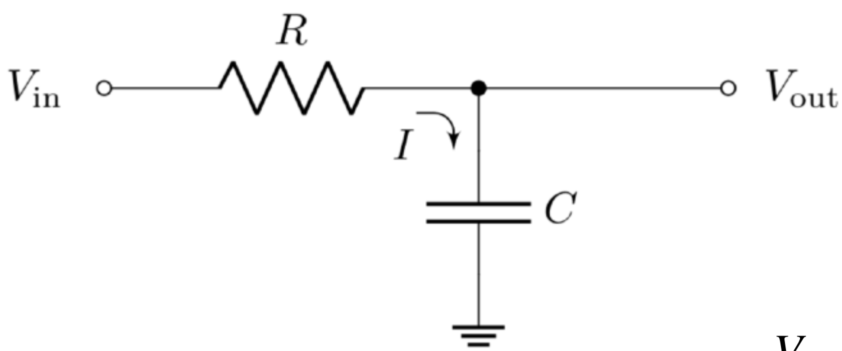

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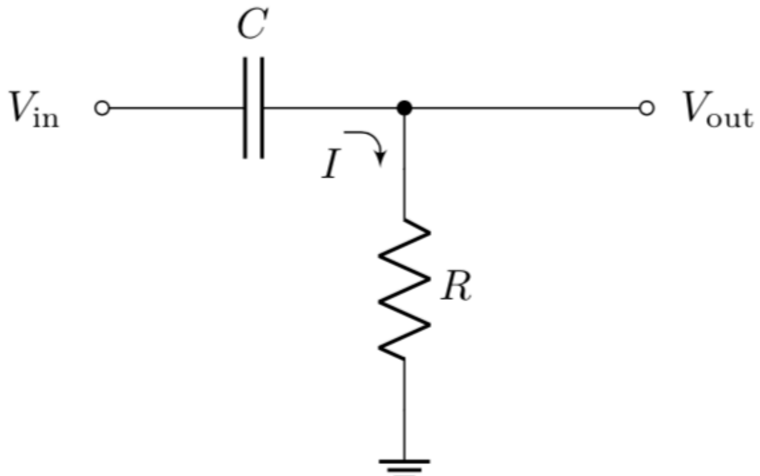


How about this one?

At high frequency, the capacitor becomes low impedance, so it shorts the input to the output.

And it blocks low frequency.

Expressing AC signals in complex notation

It simplifies the handling of AC signals to treat them with complex notation, which includes both amplitude and phase. We can do this with Euler's formula.

$e^{i\theta} = \cos\theta + i\sin\theta$

So we can express $V(t) = V_0 \cos \omega t$ as $V(t) = V_0\, e^{i\omega t}$ well the real part is $V(t)$.

We already use *i* for a small signal change in current, so in electronics we instead use $j^2 = -1$. (Textbook uses j=-i.) So we'll represent the voltage as

$$\tilde{V}(t) = V_0\, e^{j\omega t}$$

<u>The tilde reminds us that this is the complex representation.</u>
Now calculate the current from $I = C\, dV/dt$.

$$\tilde{I} = C\, d\tilde{V}/dt = C\, j\omega V_0\, e^{j\omega t} = j\omega C \tilde{V}$$

So, we can write an Ohm's law like relation between $\tilde{V}$ and $\tilde{I}$:

$$\tilde{V} = \tilde{I}\,(1/j\omega C) = \tilde{I}\,(-j/\omega C) \quad \Rightarrow \quad \tilde{V} = \tilde{I}\,\tilde{X}_C \quad \text{cf} \quad \tilde{V} = \tilde{I}\,\tilde{X}_R$$

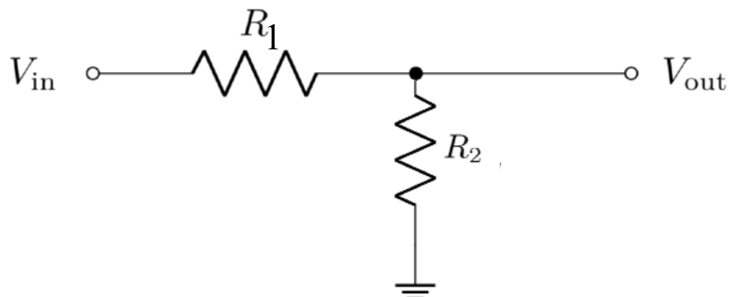
Where the impedance of the capacitor in complex representation is

$\tilde{X}_C = -j/\omega C$ cf $\tilde{X}_R = R$ (the $-j$ carries information about the phase)

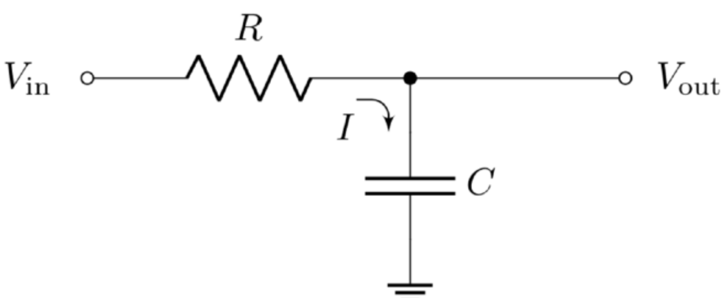
Expressing AC signals in complex notation

This will simplify handling a mix of capacitors and resistors.

This makes the circuits below just complex voltage dividers.



$$\tilde{V}_{\text{out}} = \tilde{I}R_2 = \tilde{V}_{\text{in}} \frac{R_2}{R_1+R_2} = \tilde{V}_{\text{in}} \frac{\tilde{X}_2}{\tilde{X}_1+\tilde{X}_2}$$



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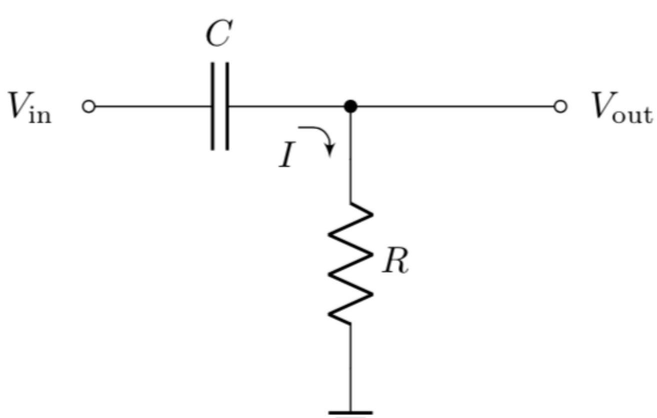

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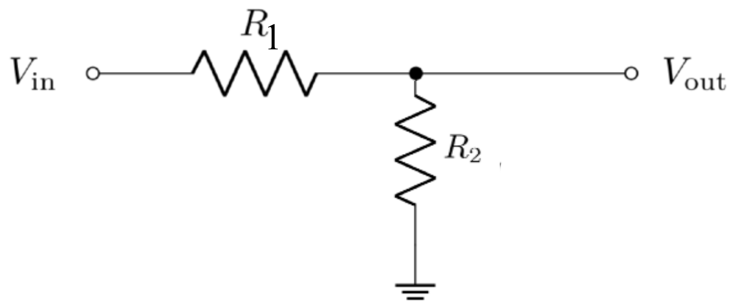

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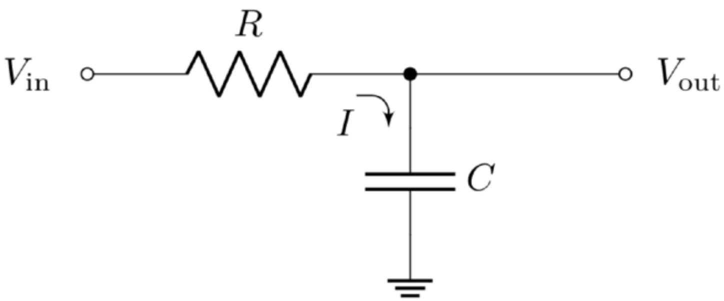

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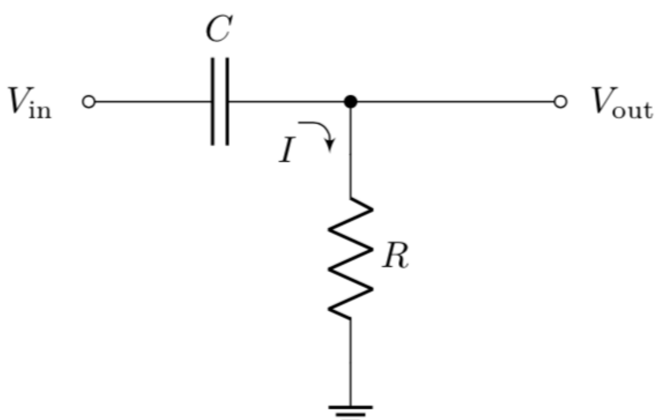
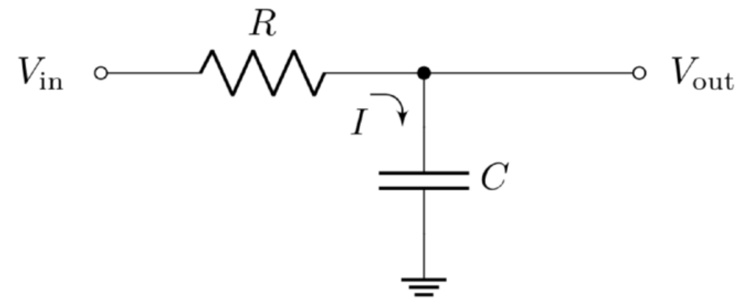


$$\tilde{V}_{out} = \tilde{I}\,\tilde{X}_R = \tilde{V}_{in}\frac{R}{R+\tilde{X}_C}$$

We can use this to calculate the output voltage in terms of input voltage, V_{out}/V_{in}.

Calculating V_{out}/V_{in} with complex impedance

$$\tilde{V}_{\text{out}} = \tilde{I}\tilde{X}_C = \frac{\tilde{V}_{\text{in}}}{R+\tilde{X}_C}\tilde{X}_C = \tilde{V}_{\text{in}}\frac{\tilde{X}_C}{R+\tilde{X}_C}$$



Let's leave the voltages as arbitrary, but put in the explicit form for $\tilde{X}_C$.

$$\tilde{V}_{\text{out}} = \tilde{V}_{\text{in}}\left[\frac{-j/\omega C}{R-j/\omega C}\right]$$

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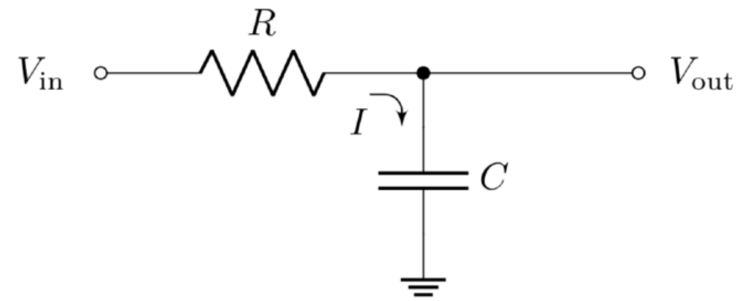

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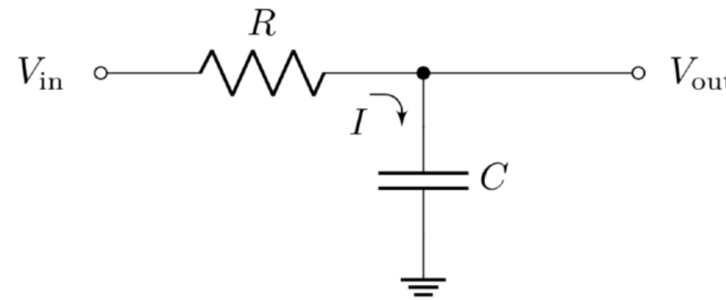

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$$\tilde{V}_{\text{out}} = \tilde{V}_{\text{in}}\frac{1/\omega^2C^2-jR/\omega C}{R^2+1/\omega^2C^2} * \left[\frac{\omega^2C^2}{\omega^2C^2}\right] = \tilde{V}_{\text{in}}\frac{1-j\omega RC}{1+\omega^2R^2C^2}$$

Calculating V_{out}/V_{in} with complex impedance

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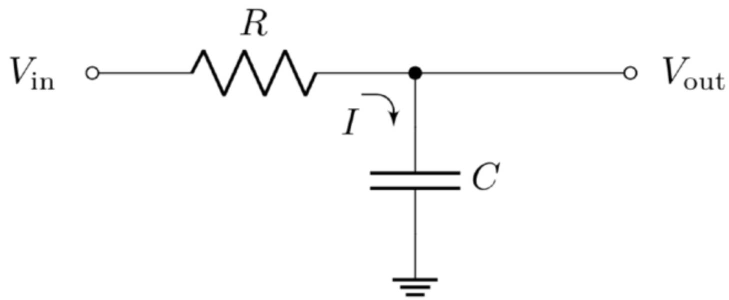

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$$|\tilde{V}_{\text{out}}| = |\tilde{V}_{\text{in}}|\left|\frac{1-j\omega RC}{1+\omega^2R^2C^2}\right|$$

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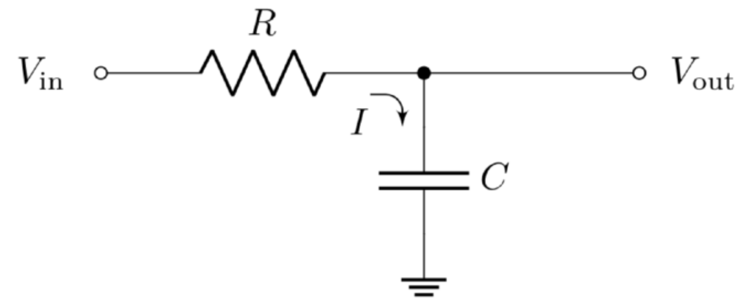

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$$\frac{|\tilde{V}_{\rm out}|}{|\tilde{V}_{\rm in}|} = \left|\frac{1-j\omega RC}{1+\omega^2R^2C^2}\right| = \sqrt{\left[\frac{1-j\omega RC}{1+\omega^2R^2C^2}\right]\left[\frac{1-j\omega RC}{1+\omega^2R^2C^2}\right]^*} = \sqrt{\left[\frac{1-j\omega RC}{1+\omega^2R^2C^2}\right]\left[\frac{1+j\omega RC}{1+\omega^2R^2C^2}\right]}$$

$$\frac{|\tilde{V}_{\rm out}|}{|\tilde{V}_{\rm in}|} = \sqrt{\frac{1+\omega^2R^2C^2}{(1+\omega^2R^2C^2)^2}} = \frac{\sqrt{1+\omega^2R^2C^2}}{1+\omega^2R^2C^2} = \frac{1}{\sqrt{1+\omega^2R^2C^2}}$$

This is the response function for the circuit, and it is frequency dependent through $\omega = 2\pi f$.

Calculating V_{out}/V_{in} with complex impedance

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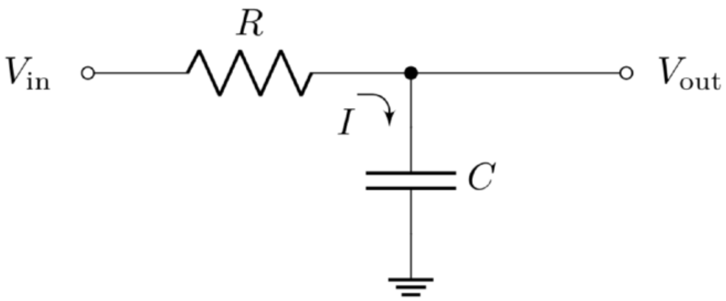

$V_{out} \rightarrow V_{in}$ as $\omega \rightarrow 0$ and $V_{out} \rightarrow 0$ as $\omega \rightarrow \infty$.

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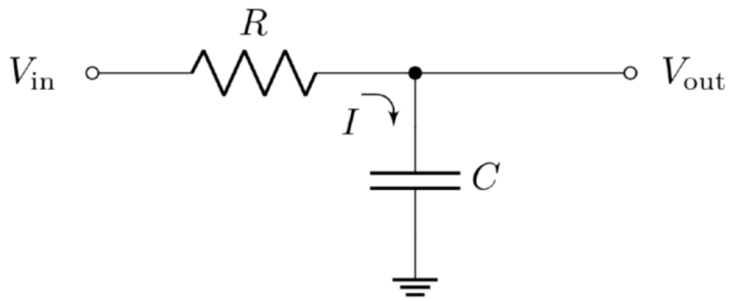

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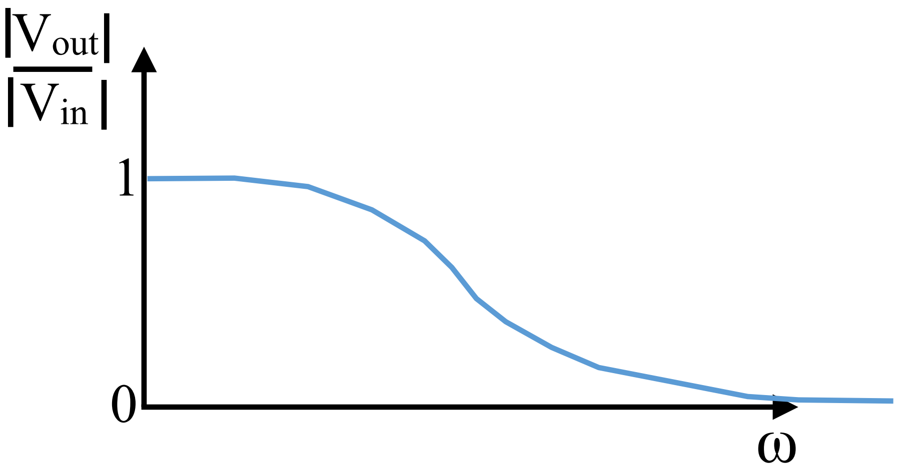

Calculating V_{out}/V_{in} with complex impedance

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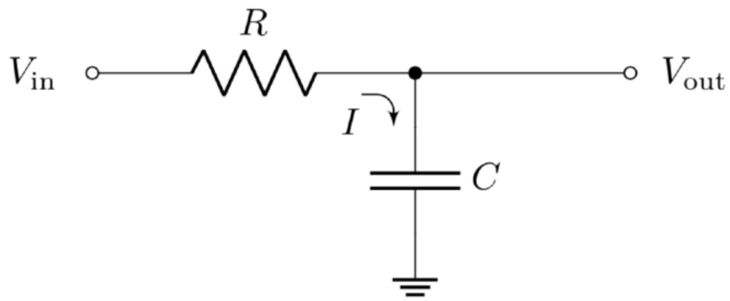

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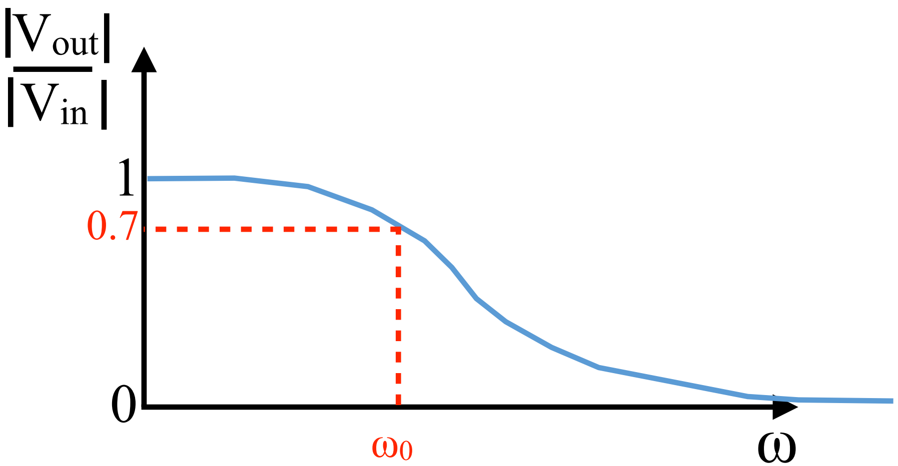


Can characterize the frequency scale with

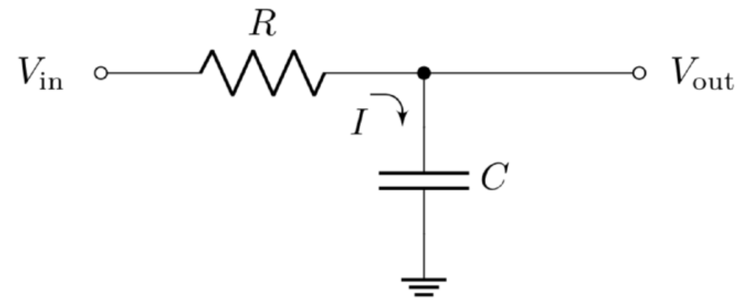
$$\omega^2 = 1/R^2C^2 \Rightarrow \omega = 1/RC$$

$$\frac{|\tilde{V}_{\rm out}|}{|\tilde{V}_{\rm in}|} = \frac{1}{\sqrt{2}} = -3 \text{ dB}$$

Call this the 3dB frequency, or the roll-off frequency, or the break point, or cut-off frequency, or just ω_0.

Calculating V_{out}/V_{in} with complex impedance

$$\frac{|\tilde{V}_{\text{out}}|}{|\tilde{V}_{\text{in}}|} = \frac{1}{\sqrt{1+\omega^2 R^2 C^2}}$$



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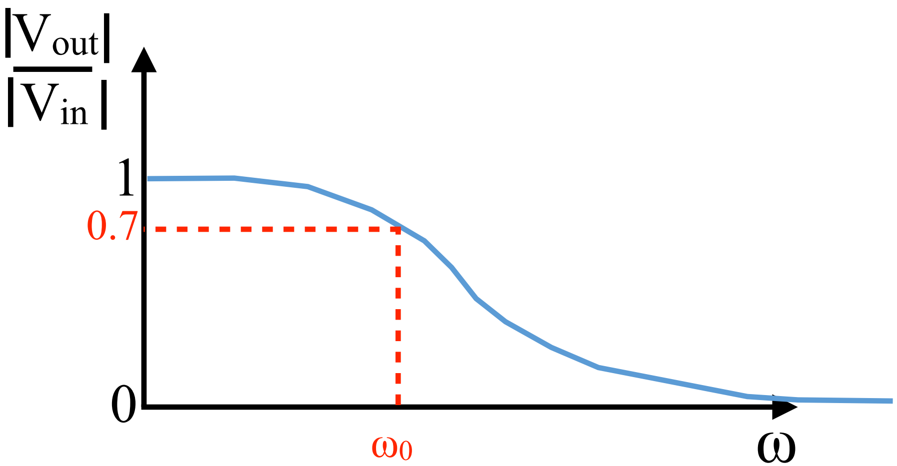

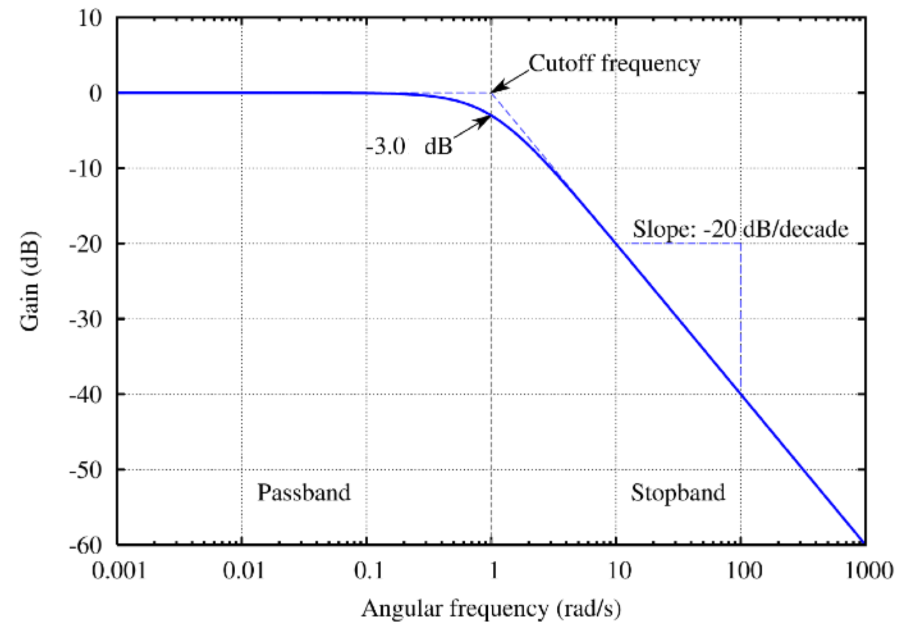

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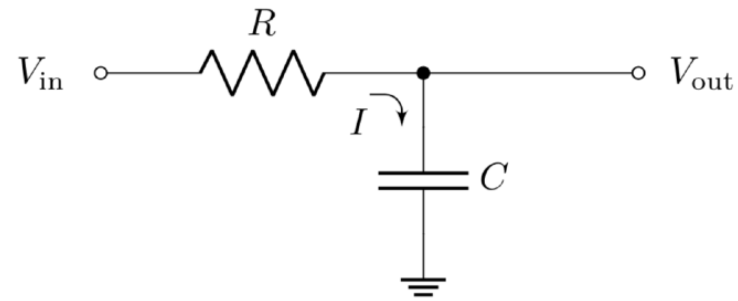

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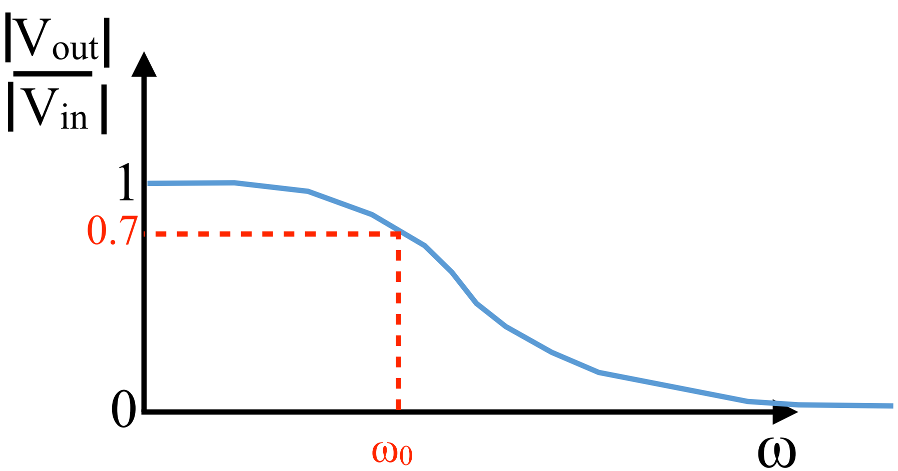

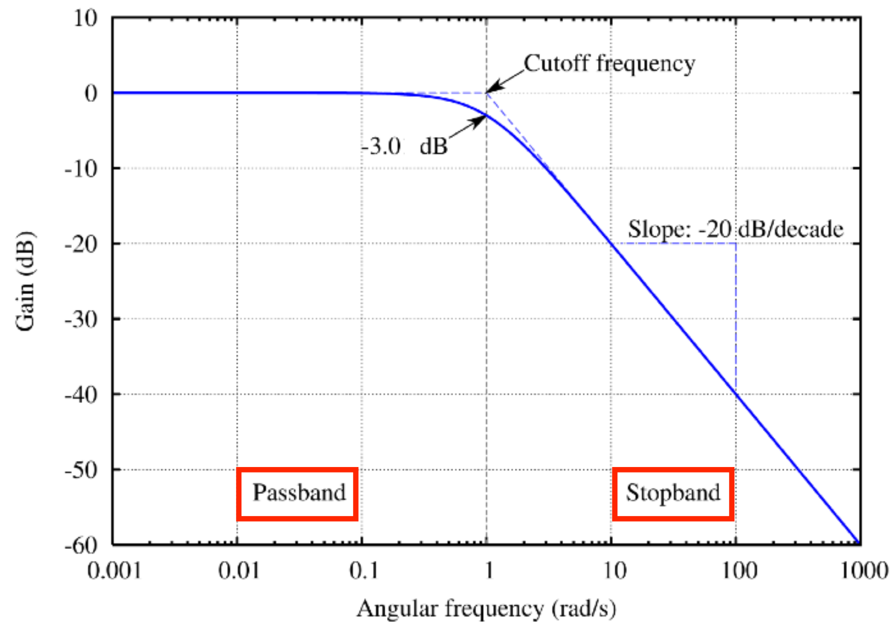

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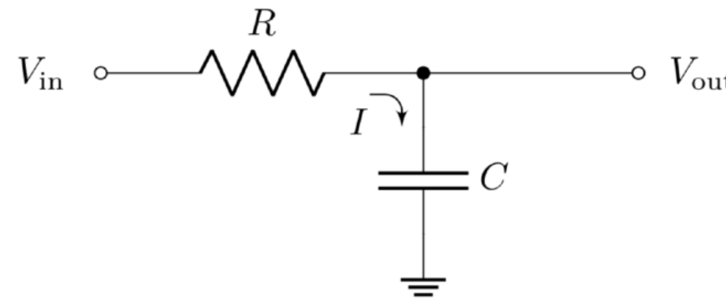

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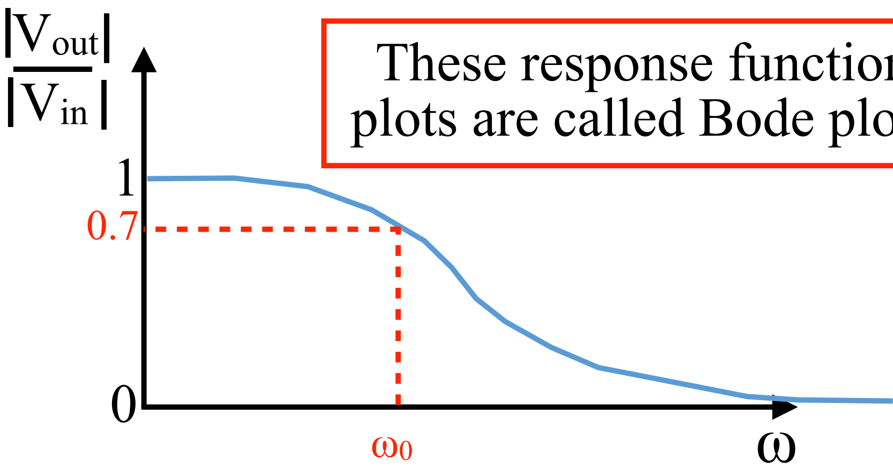

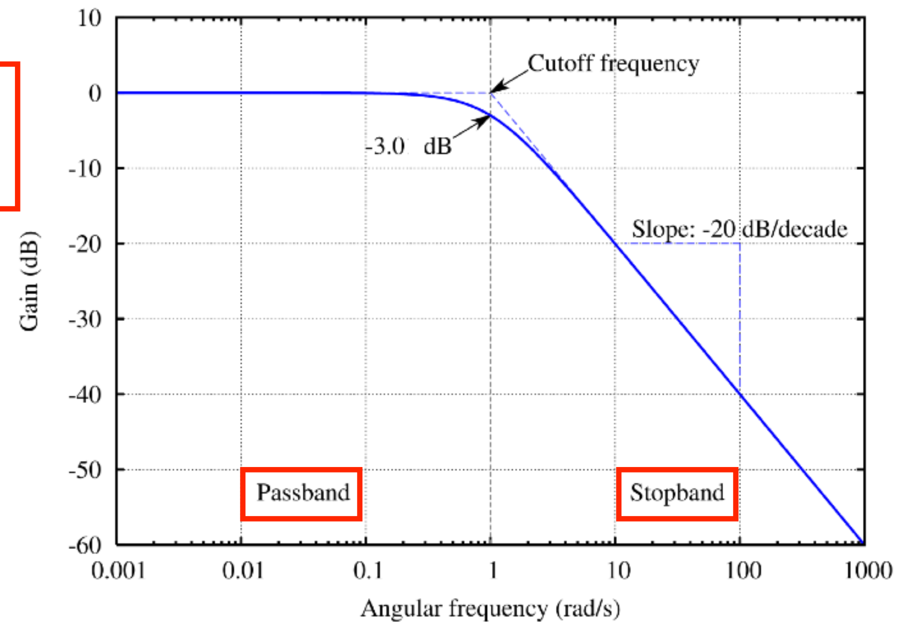

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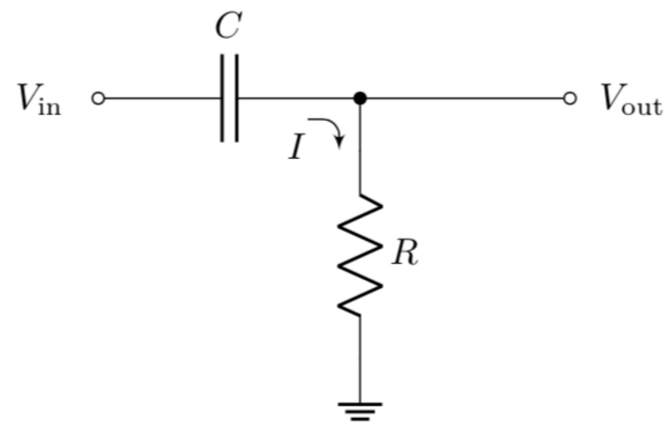

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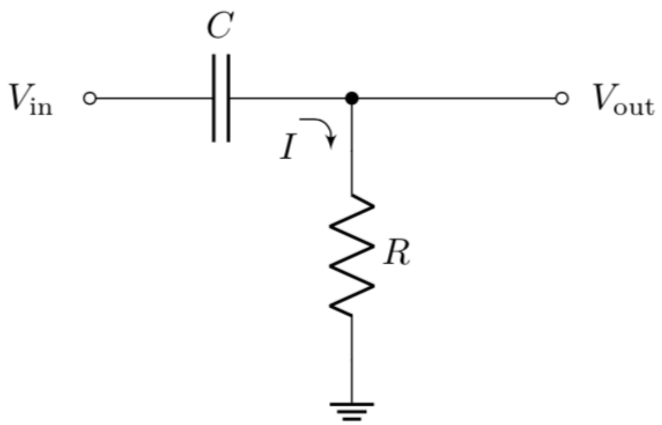

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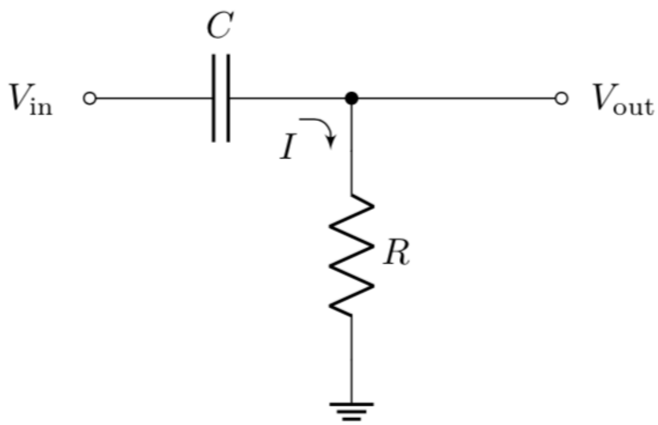

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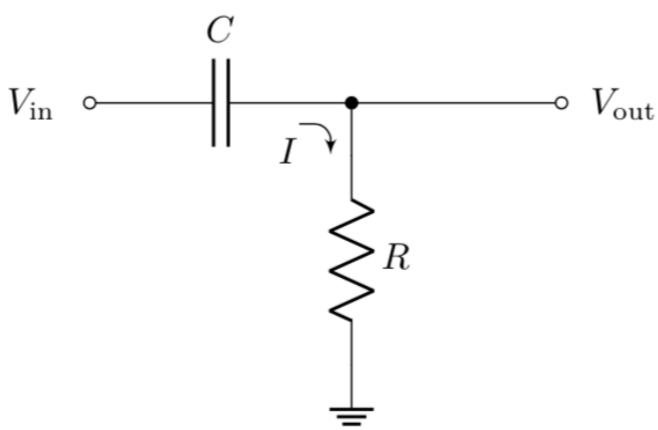

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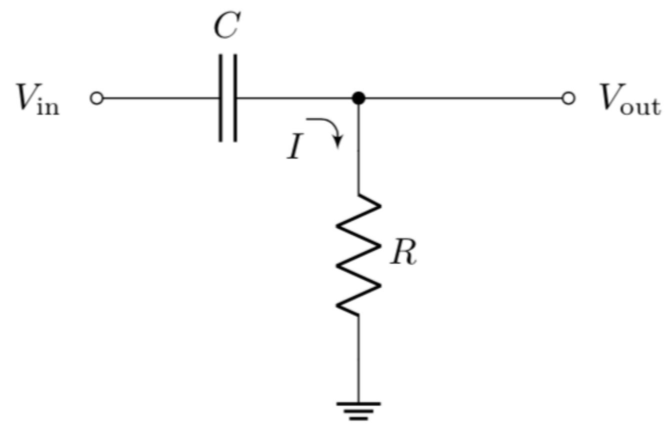

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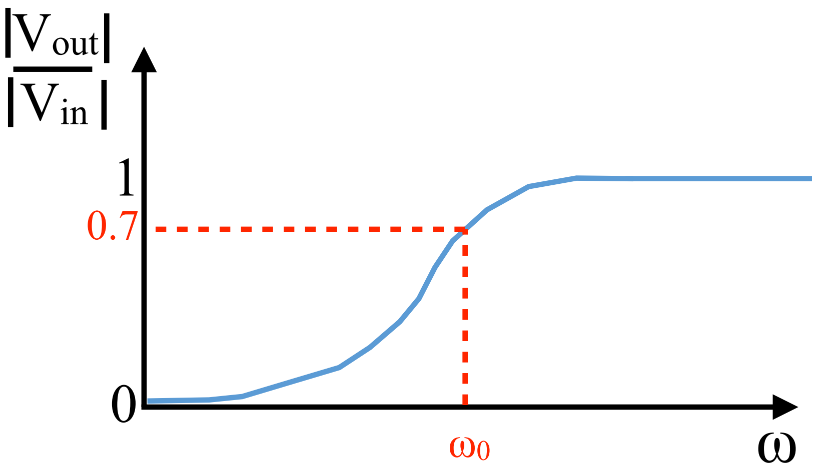

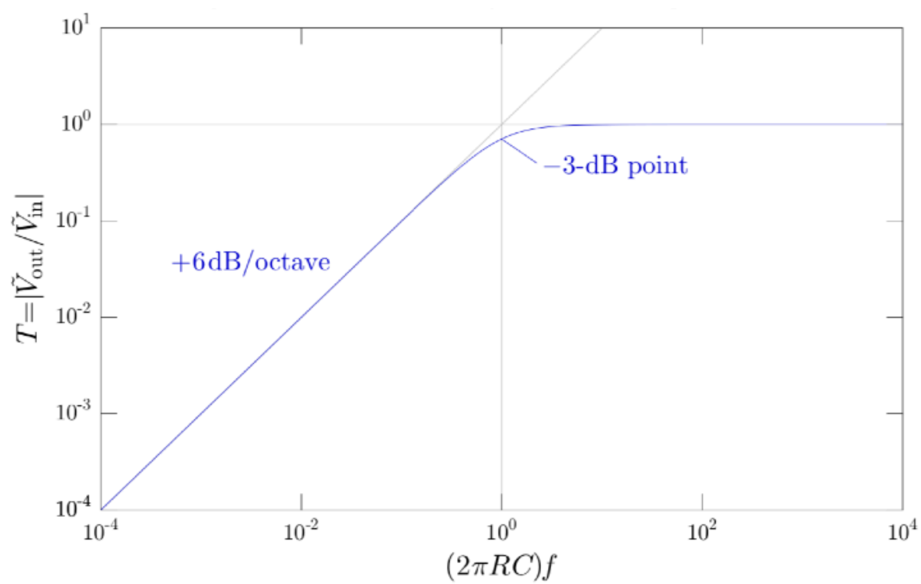

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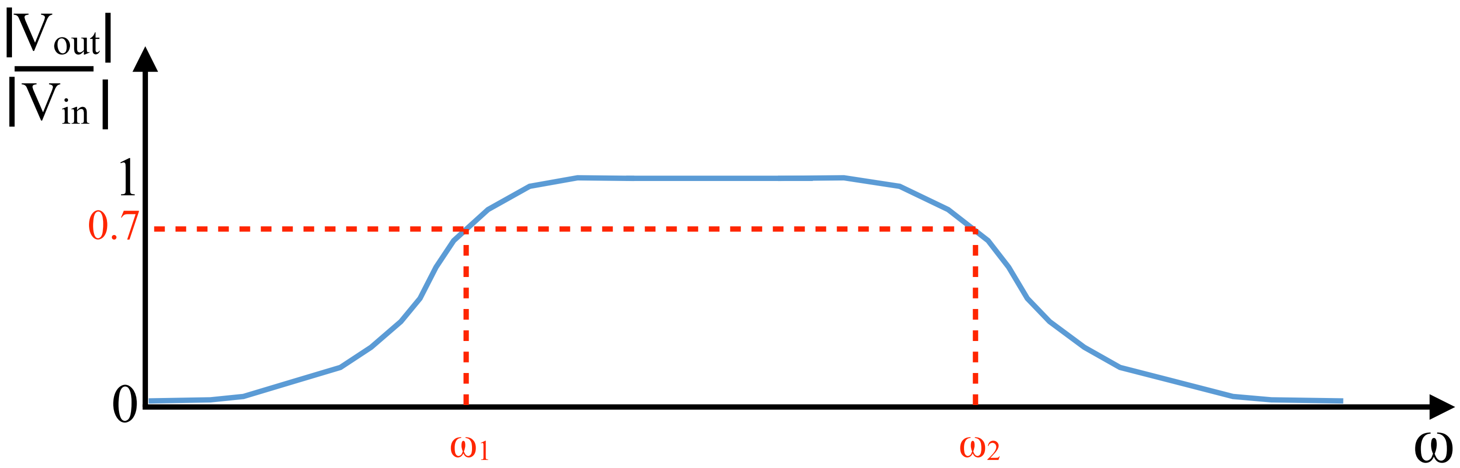
How would you make a band-pass filter?



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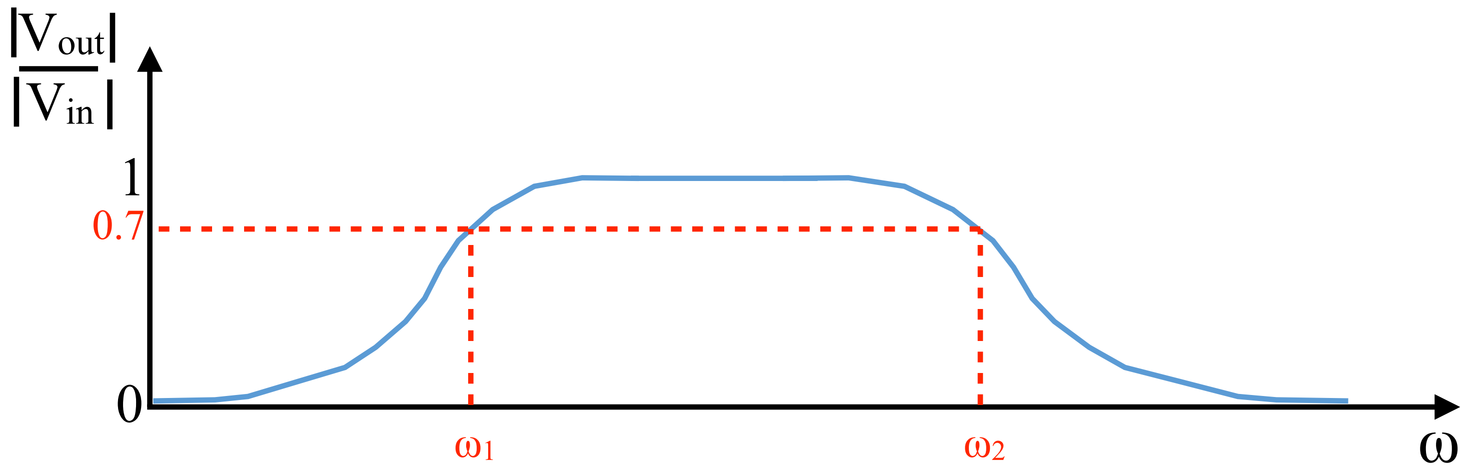
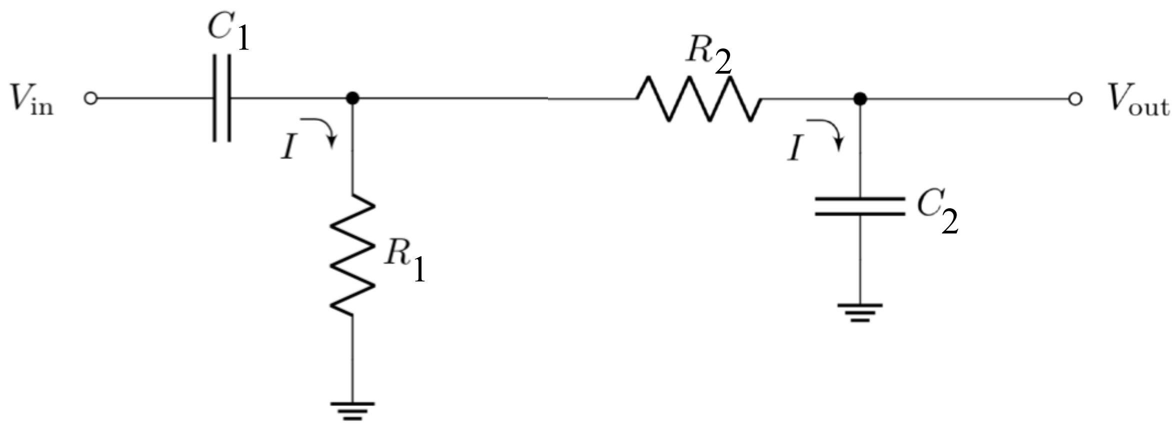




Often called a CR-RC filter.

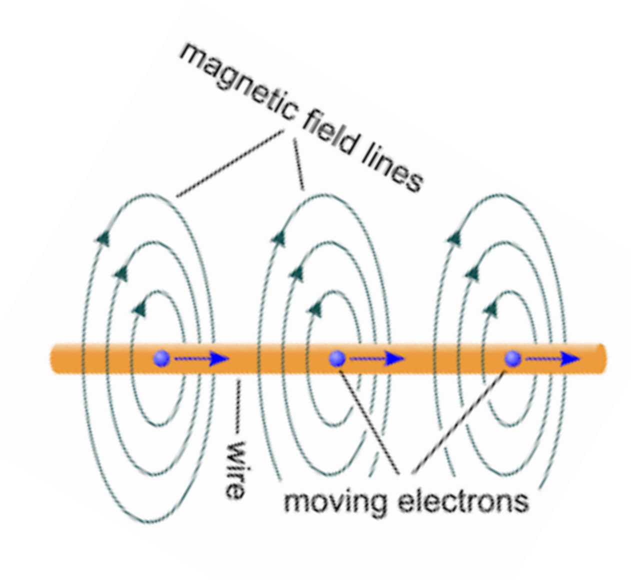
Remove DC offset and remove high frequency noise.

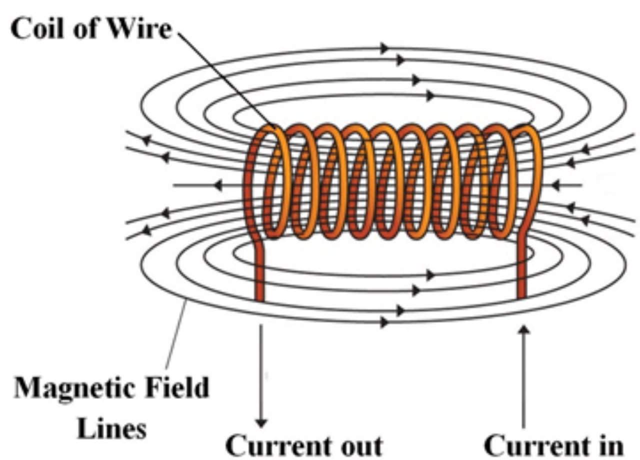
Inductors

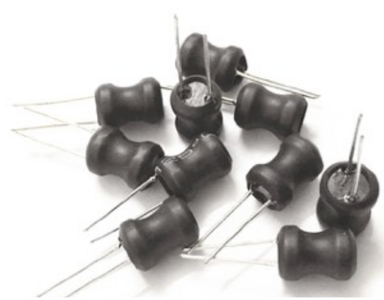
A capacitor stored energy in the electric field, in the voltage. I = C dV/dt

An inductor stores energy in the magnetic field, the current. V = L dI/dt

Any wire has inductance. A coil has more.   An iron core coil even more.



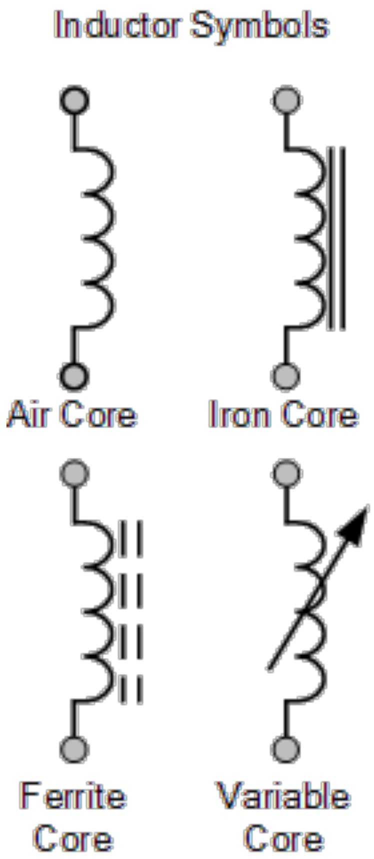




Inductors

Unit of inductance is the Henry = 1 Vs/A = Ωs (since $V = L\, dI/dt$).

Symbol looks like a coil. Some special symbols, but simple one is fine.



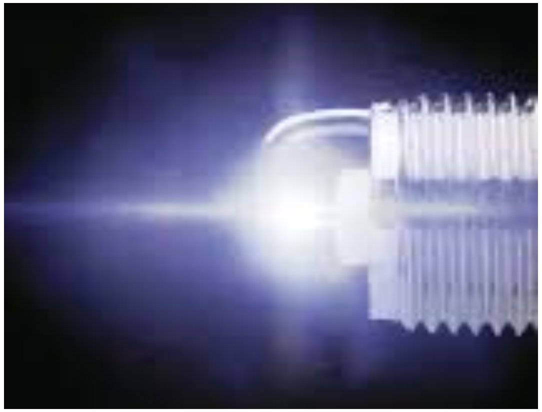
Typical values are μH to mH.

Parasitic inductance comes from any wire.
thinner and longer wires have higher L.

A 1 mm diameter wire has $L = O(1)$ nH/mm.

Inductive sparks

Since V = L dI/dt something that changes the current quickly will cause a large voltage across the inductor. A vacuum cleaner motor is mostly an inductor and will spark at the power cord if you unplug it.



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Impedance of an inductor

We can again use the complex representation of V and I to determine the impedance (reactance) of an inductor from V = L dI/dt. Represent I as

$$\tilde{I}(t) = I_0\, e^{j\omega t}$$

So the voltage is

$$\tilde{V} = L\, j\omega\, I_0\, e^{j\omega t} = j\omega L \tilde{I}$$

So, we can write an Ohm's law like relation between V and I:

$$\tilde{V} = \tilde{I}\,(j\omega L) \quad \Rightarrow \quad \tilde{V} = \tilde{I}\,\tilde{X}_L$$

where the impedance of the inductor is

$$\tilde{X}_L = j\omega L \quad \text{cf} \quad \tilde{X}_R = R \quad \text{and} \quad \tilde{X}_C = -j/\omega C$$

Different frequency dependence from capacitor.

Sign difference corresponds to a different phase response

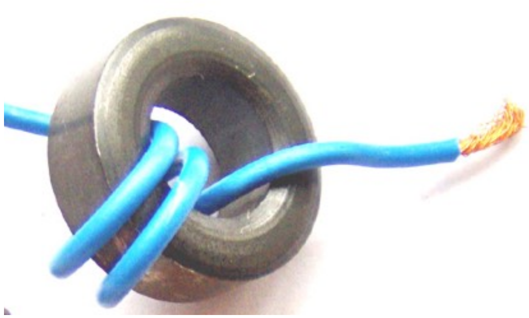
Impedance of an inductor

$\tilde{X}_L = j\omega L$

Low impedance at DC; just the normal resistance that contributes.

Impedance goes to infinity at high frequency.

- Keep wires short in high frequency circuits.
- Can block high frequency noise with a simple inductor (ferrite bead).

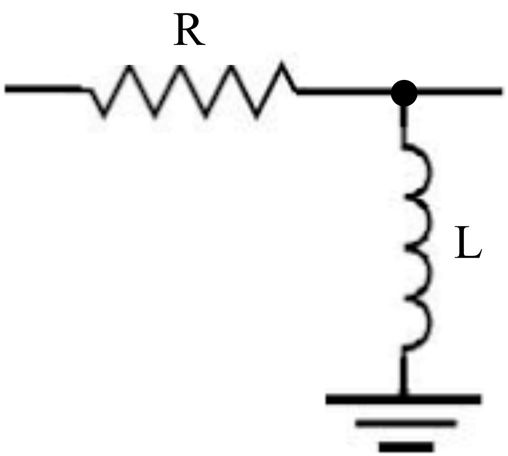


Using an inductor in a filter

$$\tilde{X}_L = j\omega L$$

Let's calculate the transfer function for an RL filter.
But first, is it high or low pass?

Hint: Think of the voltage divider equation.



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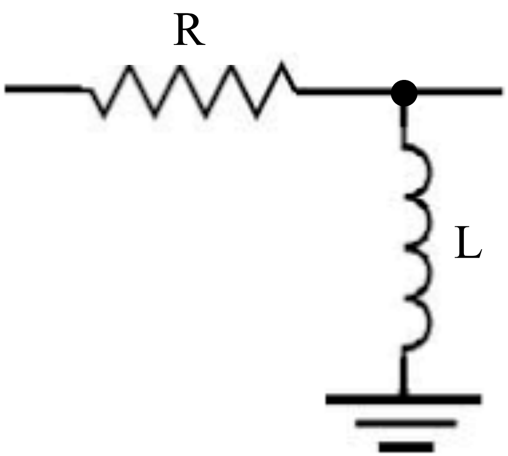

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At low ω, $X_L \rightarrow 0$ so $V_{out} \rightarrow 0$

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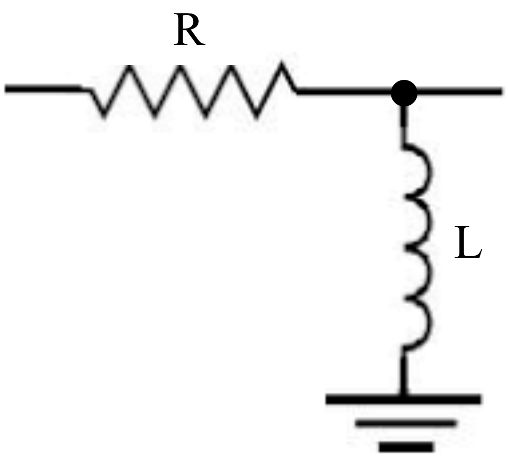

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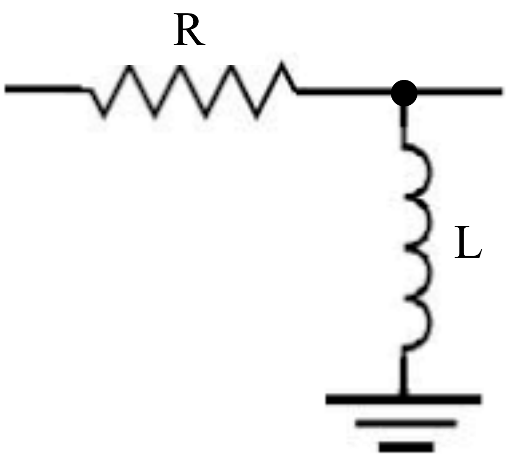

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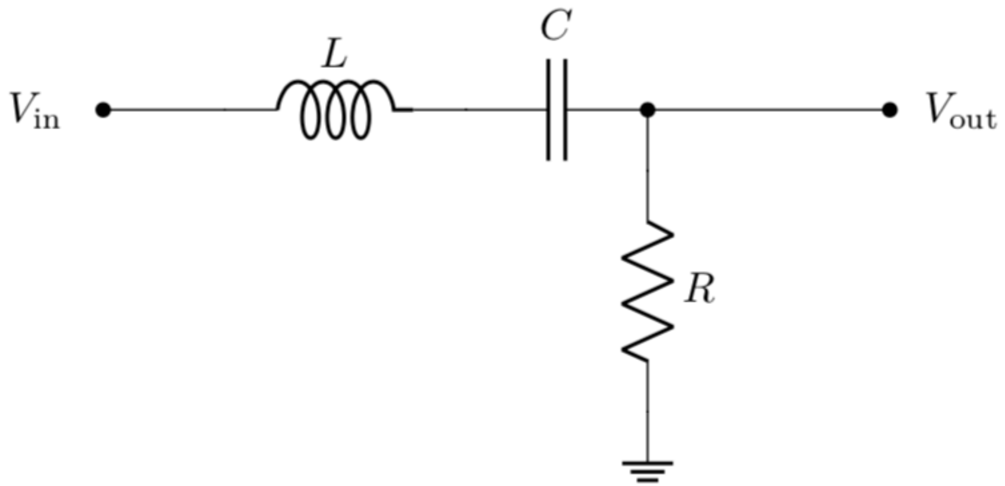
Compare to differentiator:

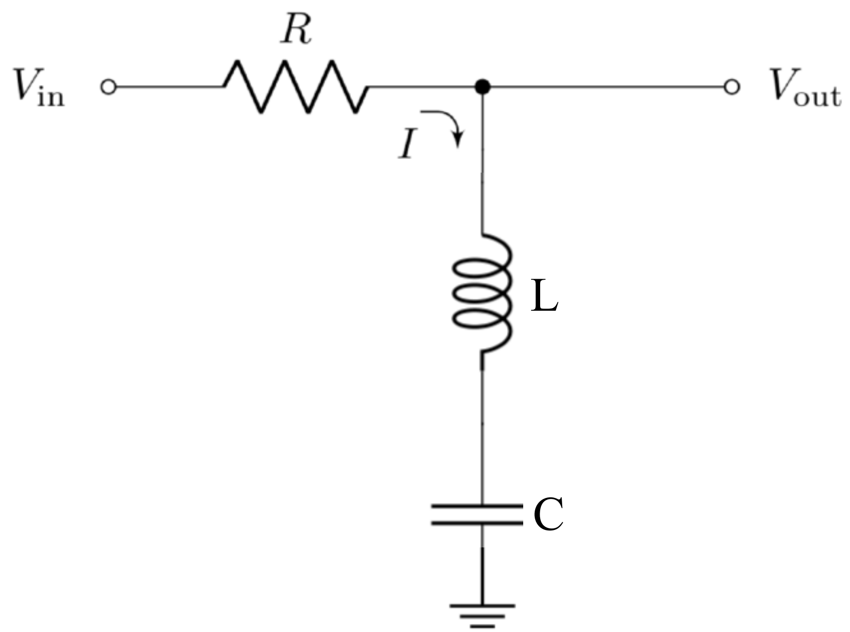
$$\frac{|\tilde{V}_{\text{out}}|}{|\tilde{V}_{\text{in}}|} = \frac{1}{\sqrt{1+1/\omega^2 R^2 C^2}} = \frac{\omega RC}{\sqrt{1+\omega^2 R^2 C^2}}$$

Combining inductors and capacitors

Textbook does this one:



I will do this one:



Combining inductors and capacitors

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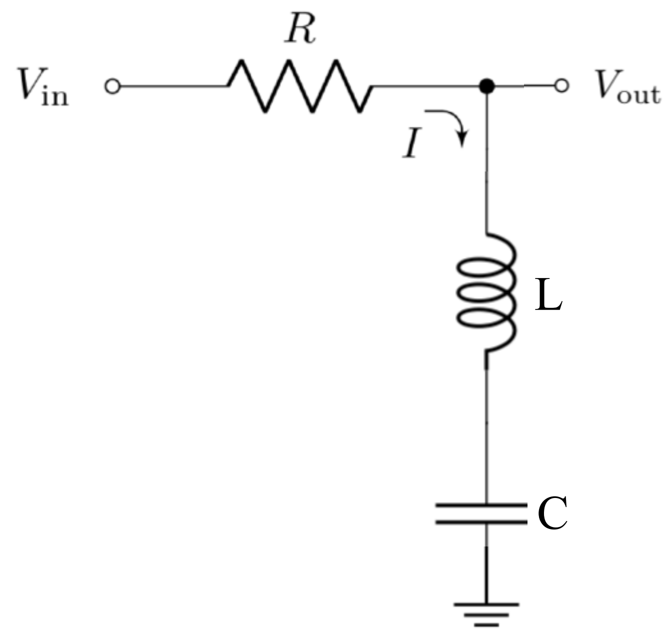

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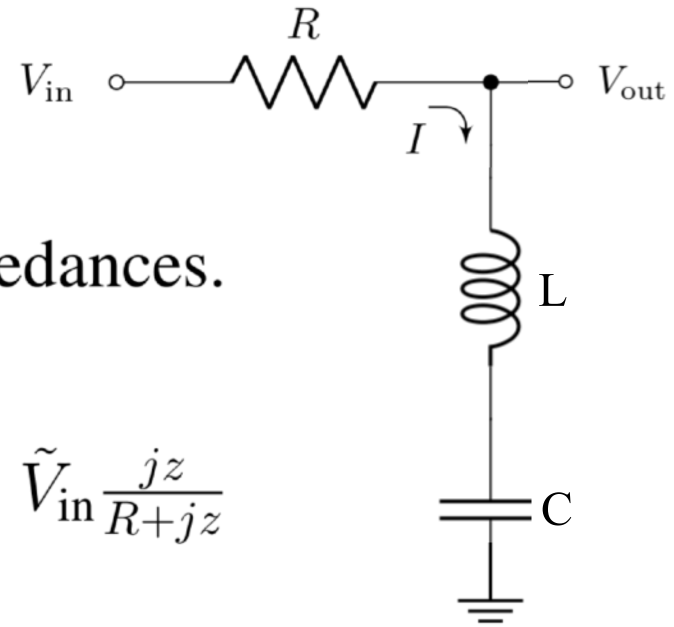

First we need to know how to combine impedances.
They all add like resistance.

$$\tilde{V}_{\text{out}} = \tilde{V}_{\text{in}} \frac{j\omega L - j/\omega C}{R + j\omega L - j/\omega C} = \tilde{V}_{\text{in}} \frac{j(\omega L - 1/\omega C)}{R + j(\omega L - 1/\omega C)} = \tilde{V}_{\text{in}} \frac{jz}{R + jz}$$

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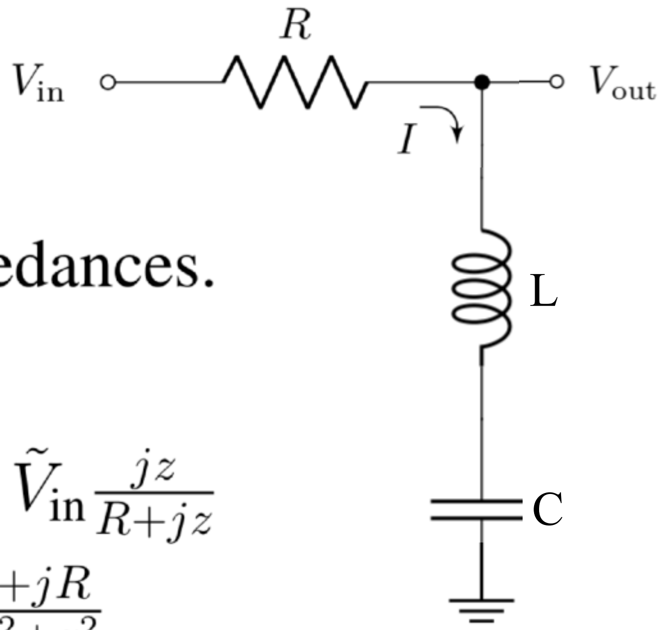

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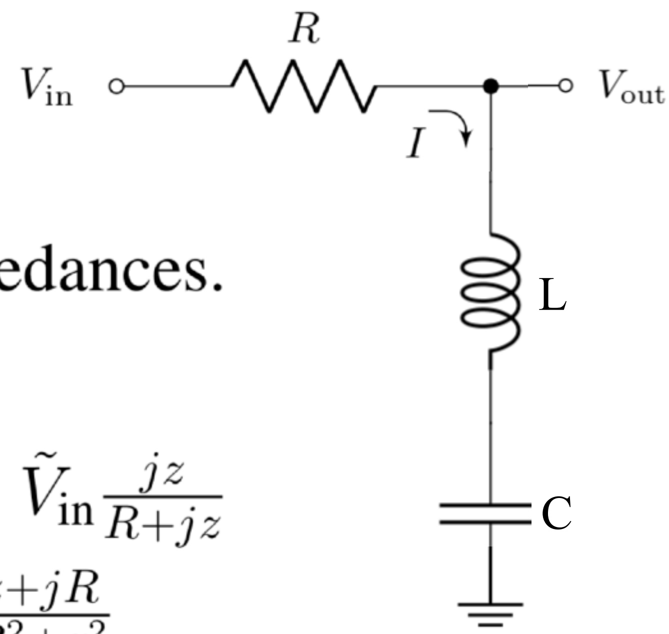
$$\frac{|\tilde{V}_{\text{out}}|}{|\tilde{V}_{\text{in}}|} = z\sqrt{\frac{z^2+R^2}{(R^2+z^2)^2}} = \frac{z}{\sqrt{z^2+R^2}} = \frac{1}{\sqrt{1+R^2/z^2}}$$

$$\frac{|\tilde{V}_{\text{out}}|}{|\tilde{V}_{\text{in}}|} \to 0 \text{ when } z = \omega L - 1/\omega C \to 0$$

$$\Rightarrow \omega_0 = 1/\sqrt{LC}$$

Combining inductors and capacitors



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They all add like resistance.

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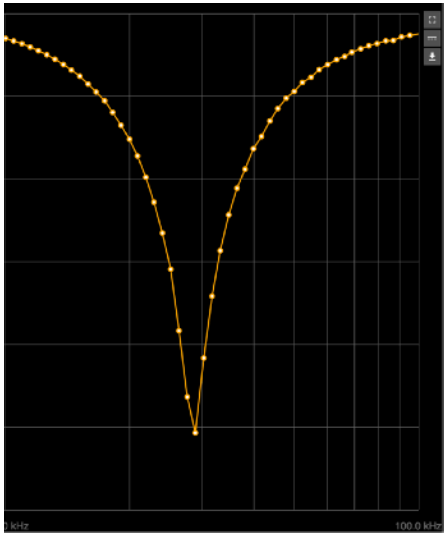
$$\tilde{V}_{\mathrm{out}} = \tilde{V}_{\mathrm{in}}\frac{jz}{R+jz}\left[\frac{R-jz}{R-jz}\right] = \tilde{V}_{\mathrm{in}}\frac{z^2+jRz}{R^2+z^2} = \tilde{V}_{\mathrm{in}}z\frac{z+jR}{R^2+z^2}$$

$$\frac{|\tilde{V}_{\mathrm{out}}|}{|\tilde{V}_{\mathrm{in}}|} = z\sqrt{\frac{z^2+R^2}{(R^2+z^2)^2}} = \frac{z}{\sqrt{z^2+R^2}} = \frac{1}{\sqrt{1+R^2/z^2}}$$



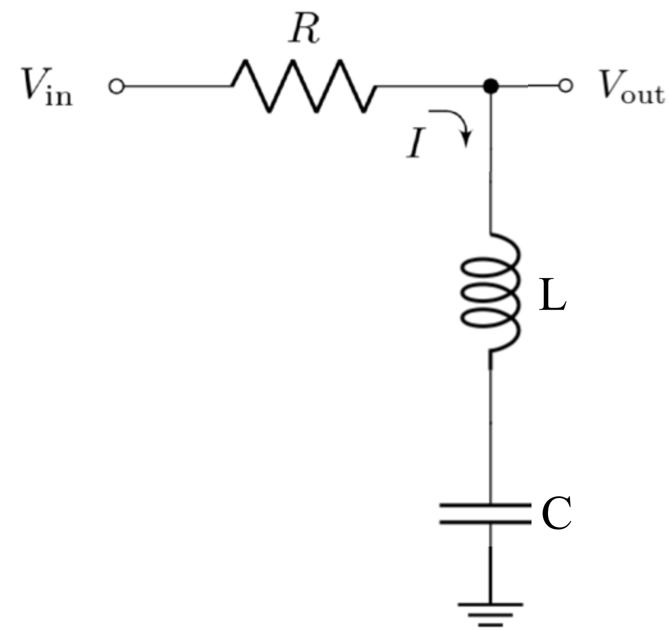
$$\frac{|\tilde{V}_{\mathrm{out}}|}{|\tilde{V}_{\mathrm{in}}|} \rightarrow 0 \text{ when } z = \omega L - 1/\omega C \rightarrow 0$$

$$\Rightarrow \omega_0 = 1/\sqrt{LC}$$

Combining inductors and capacitors

We could have deduced this more simply.

The V_{out} goes to zero when the impedance of the LC leg goes to zero. That is when it shorts the output to ground.



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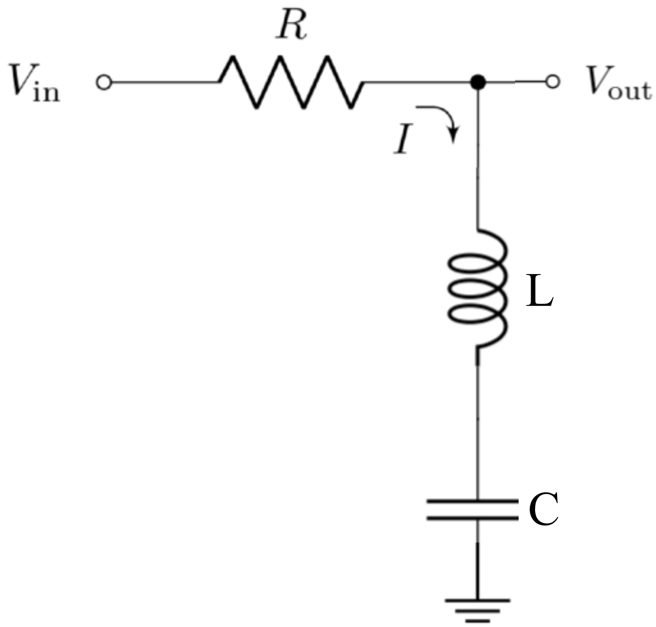

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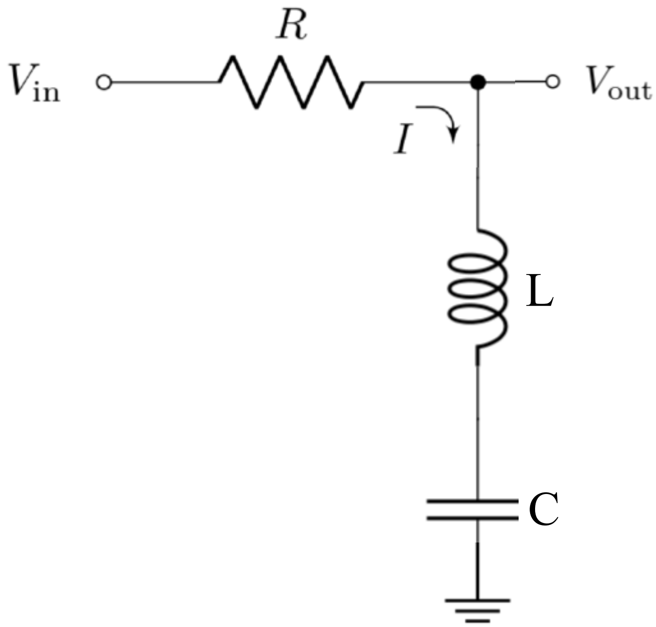

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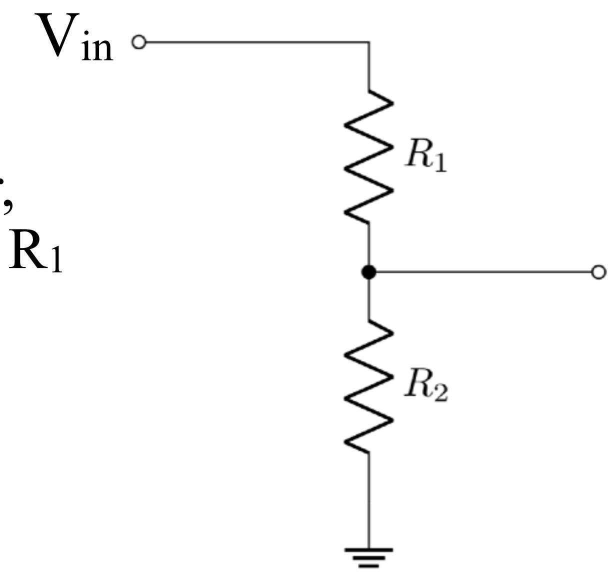
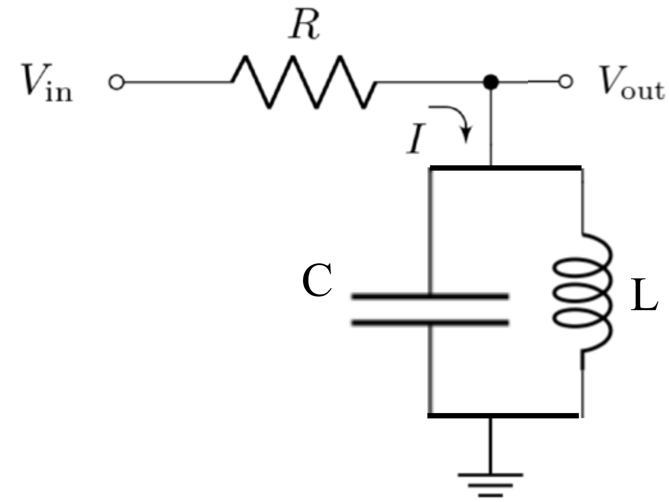


This is the same as a resistive voltage divider, smaller R_2 makes V_{out} go to zero and smaller R_1 makes V_{out} go to V_{in}.

Combining inductors and capacitors

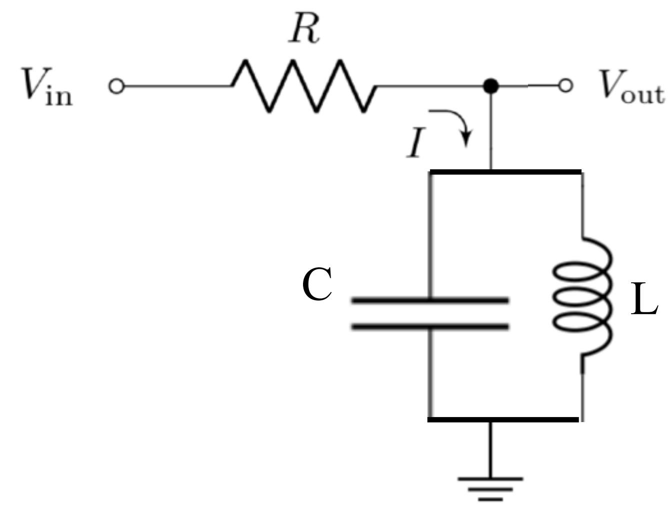
What about this one?



Combining inductors and capacitors

What about this one?

Maximized when C||L is largest.



Combining inductors and capacitors

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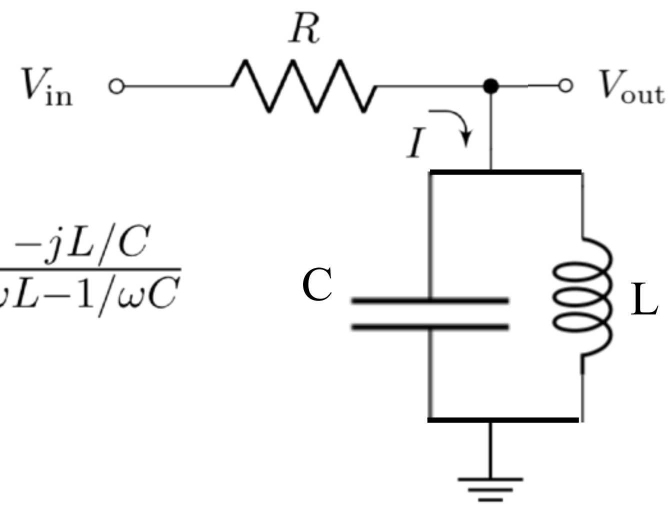


Maximized when C||L is largest.

$$\tilde{X}_{\mathrm{L}||\mathrm{C}} = \frac{\tilde{X}_{\mathrm{L}}\tilde{X}_{\mathrm{C}}}{\tilde{X}_{\mathrm{L}}+\tilde{X}_{\mathrm{C}}} = \frac{(j\omega L)(-j/\omega C)}{j\omega L - j/\omega C} = \frac{L/C}{j(\omega L - 1/\omega C)} = \frac{-jL/C}{\omega L - 1/\omega C}$$

Maximized at $\omega_0 = 1/\sqrt{LC}$

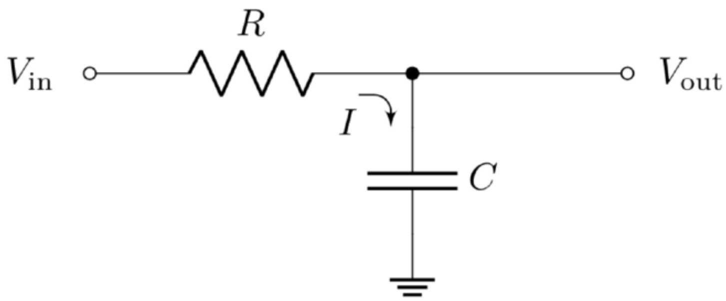
Resonance at that frequency.

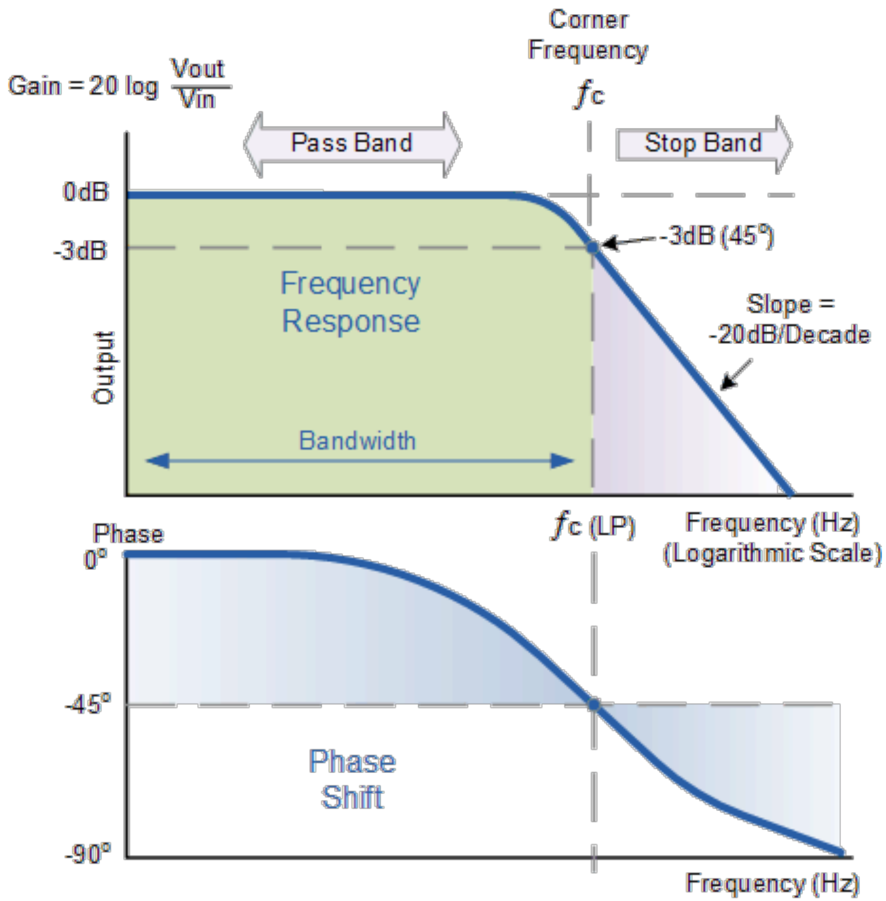
Phase

We saw that current was out of phase with voltage in a capacitor.

So we expect V_{out} and V_{in} to be out of phase.

There is a phase version of the Bode plot.





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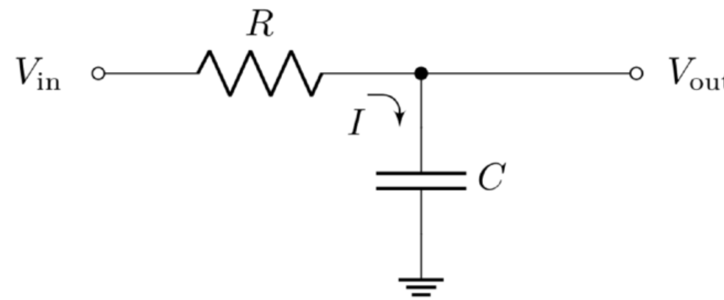
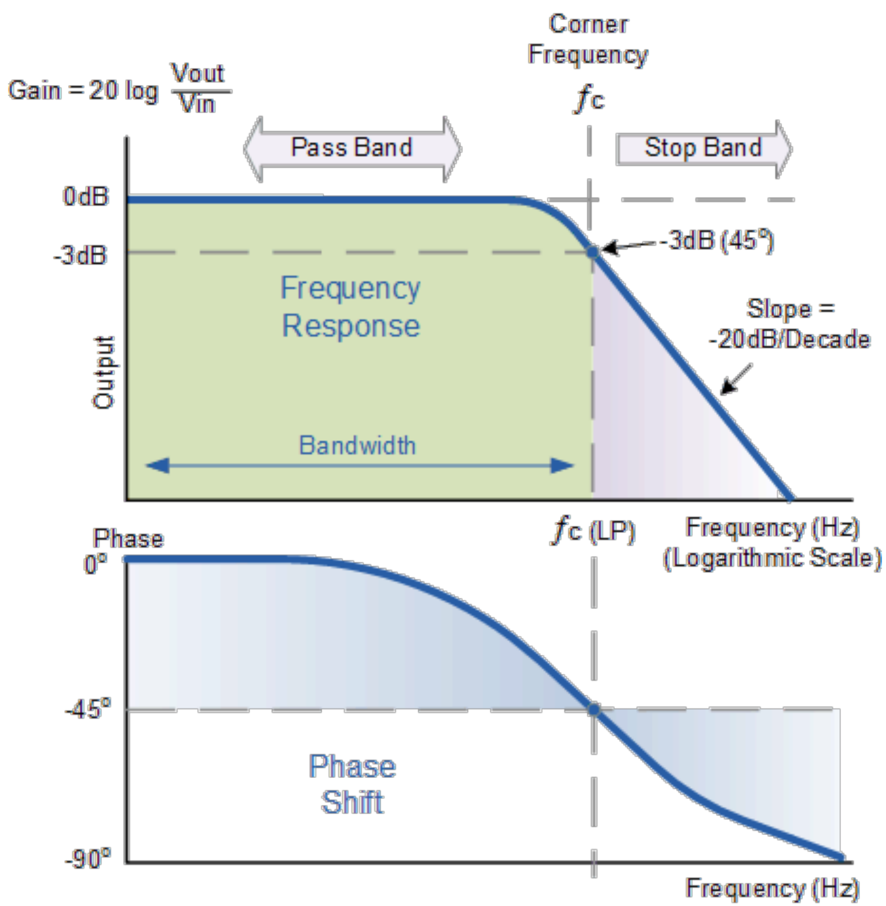
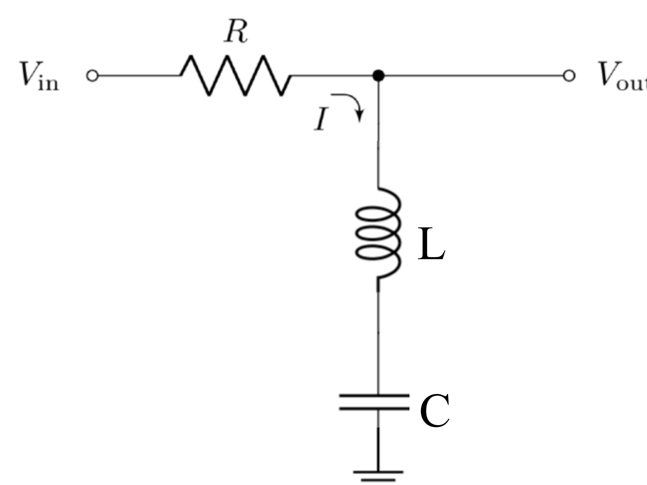




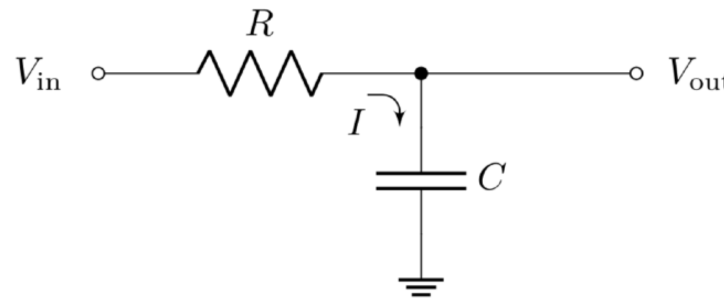
How do we calculate that shift?

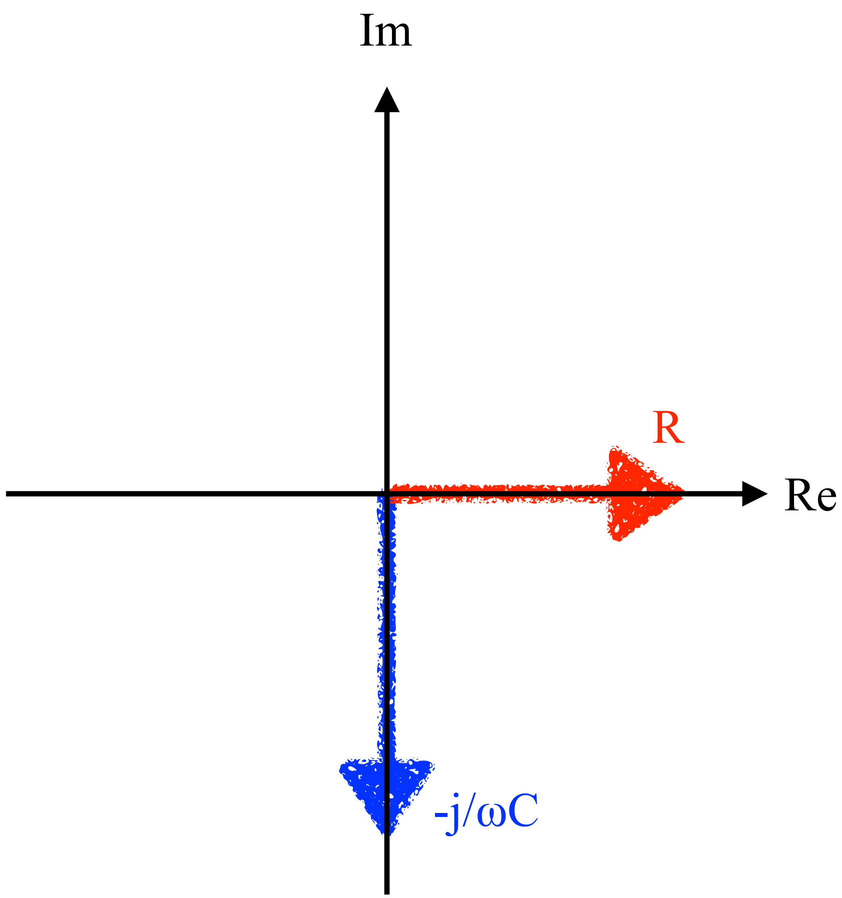
What about more complex circuits like this?



Phasor diagram

Since the impedance is complex we can draw them in the complex plane.





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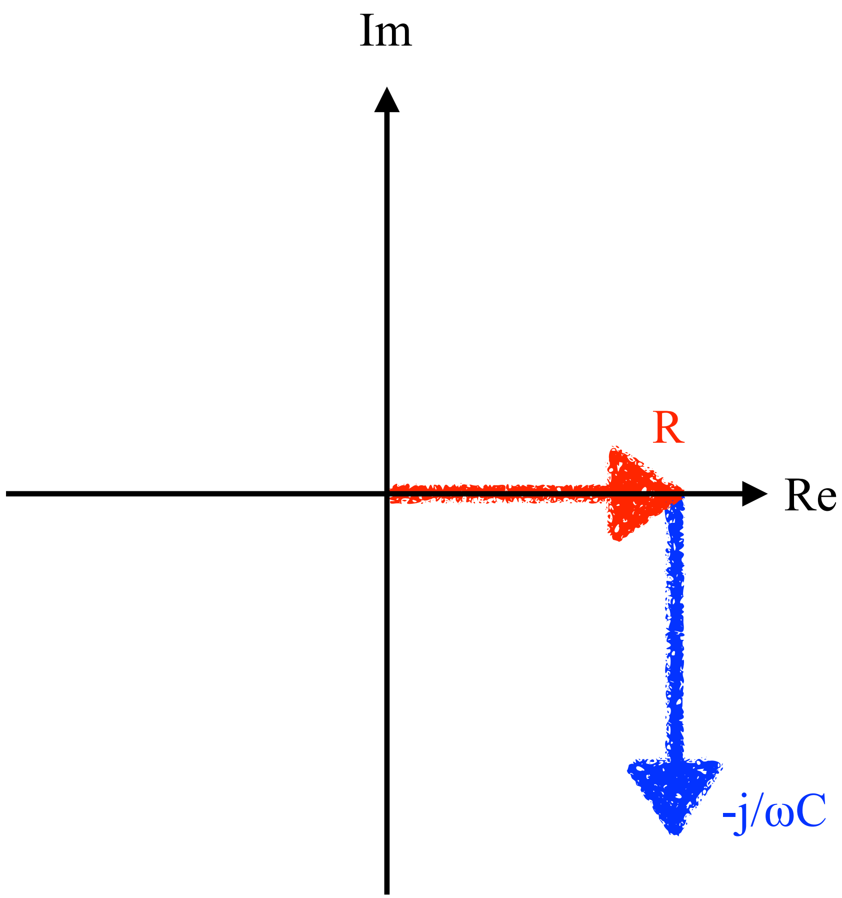

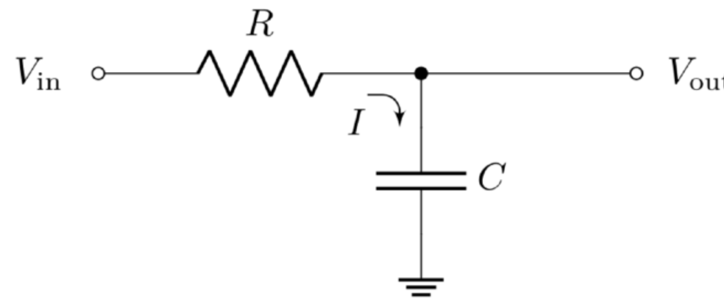

The total impedance is the sum of the R and C contributions, and they add like vectors in the complex plane.

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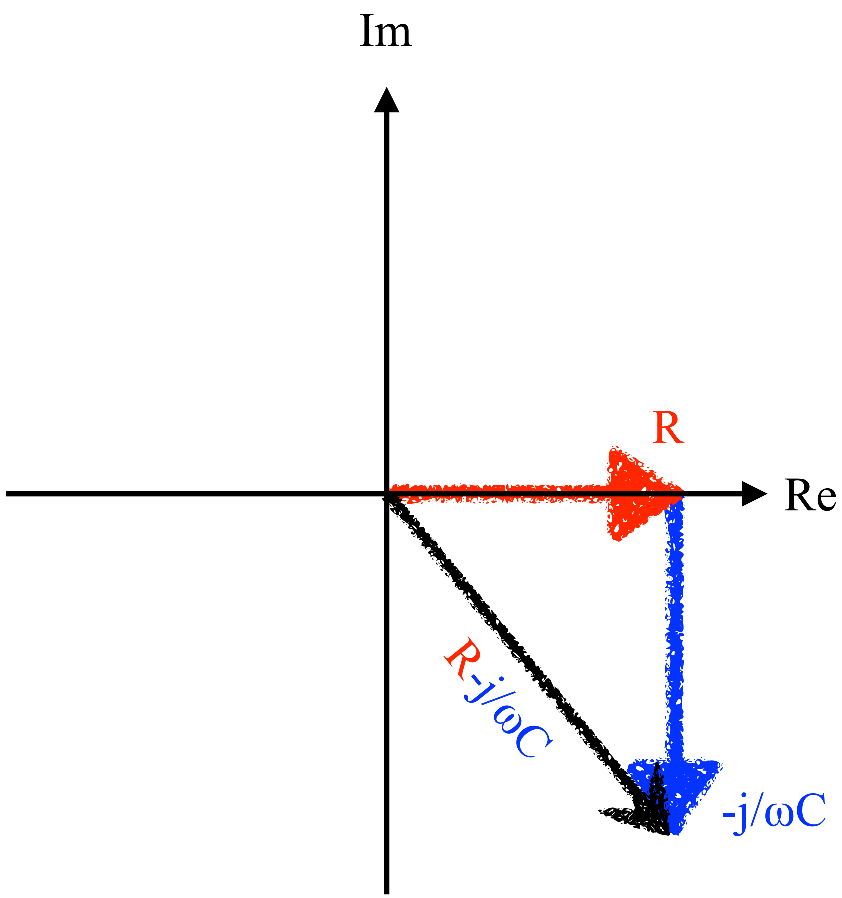

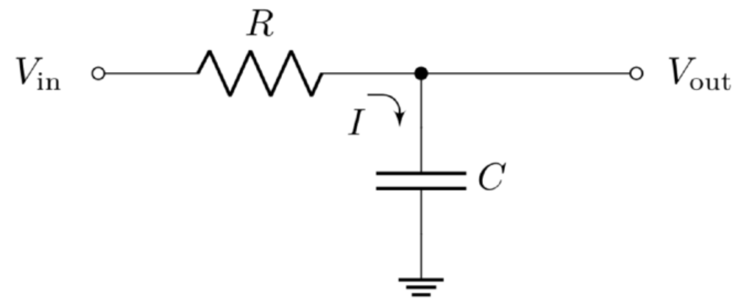

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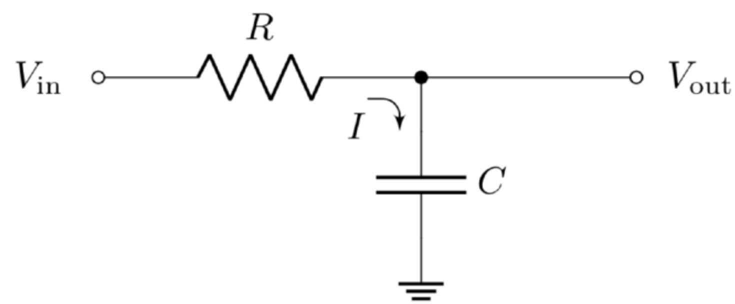

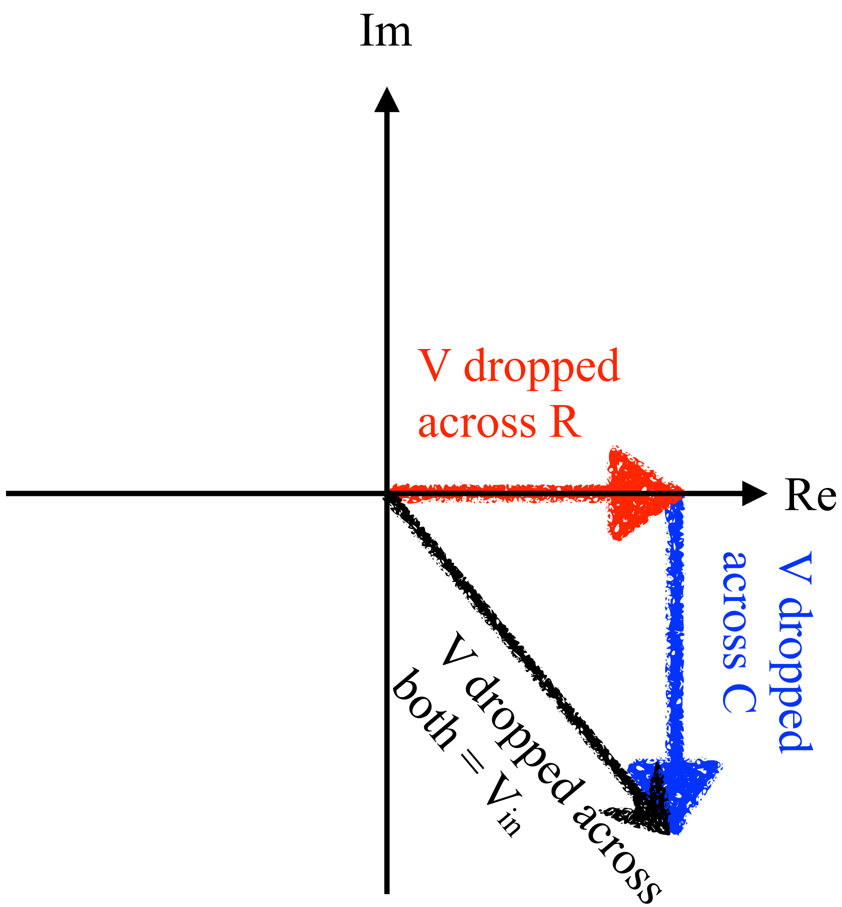

A current flowing through these impedances will drop voltages across them, so this also represents the voltage across these components.

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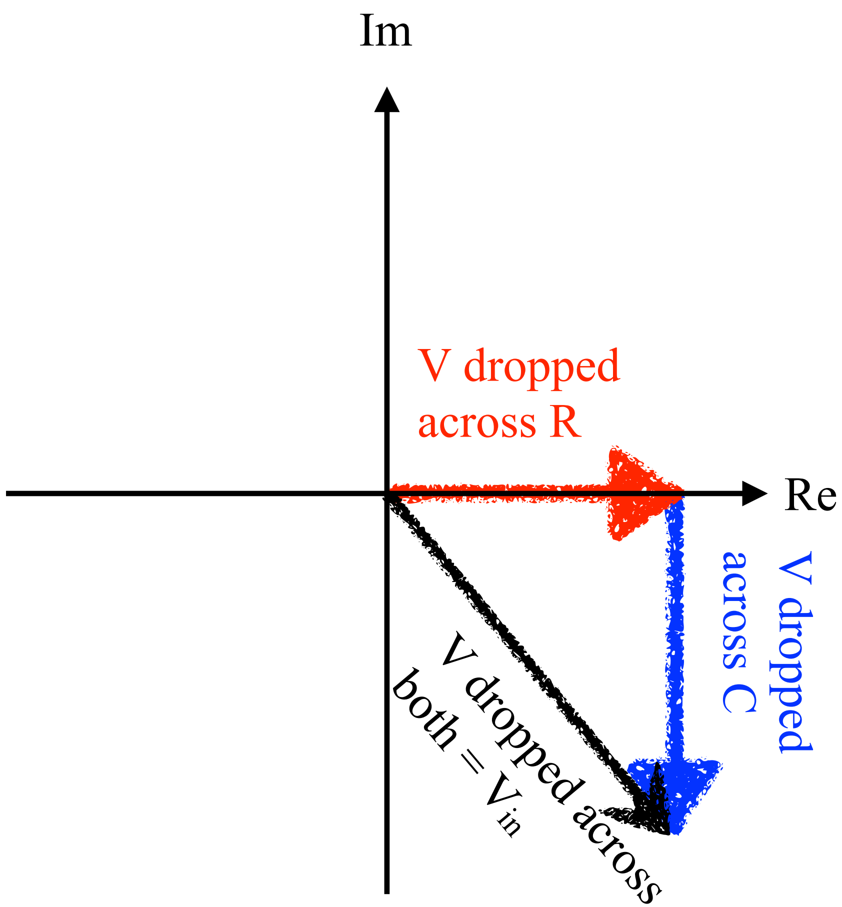

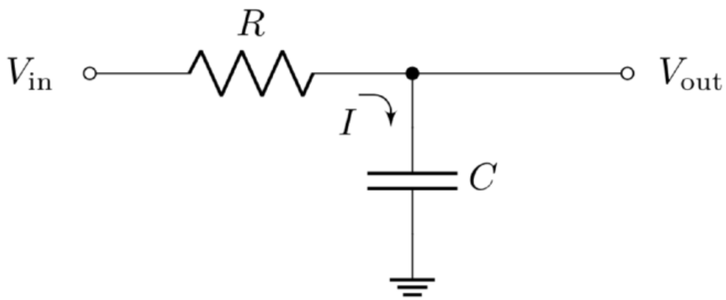

The angle between the vectors is their relative phase.

In $V_0 e^{j\omega t}$ the time dependence corresponds to sweeping the phase around.

Here we care about *relative* phase between voltage dropped across components.

Phasor diagram

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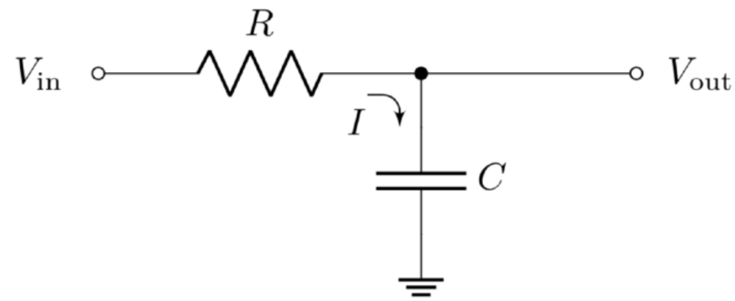

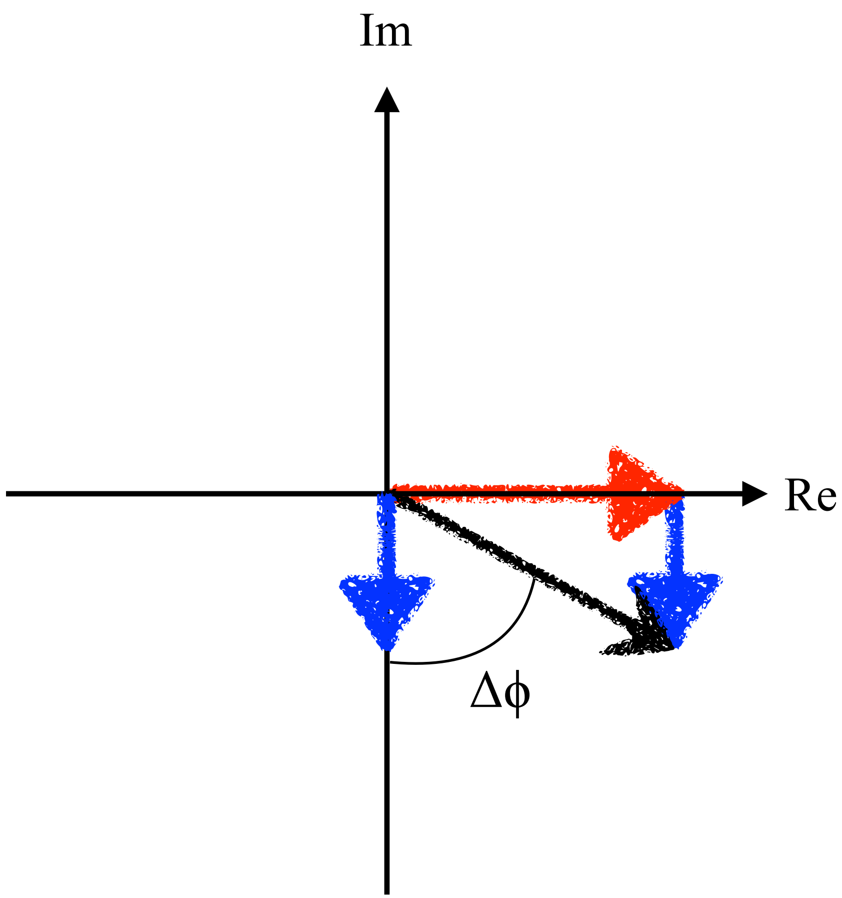

Increasing ω makes $|X_C| = 1/\omega C$ smaller. $V_C \propto 1/\omega$.

Smaller voltage drop across C has larger angle between the C vector, which is V_{out}, and the total vector, which is V_{in}.

Phasor diagram

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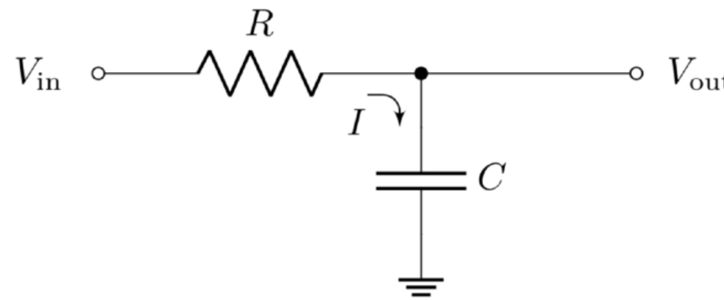

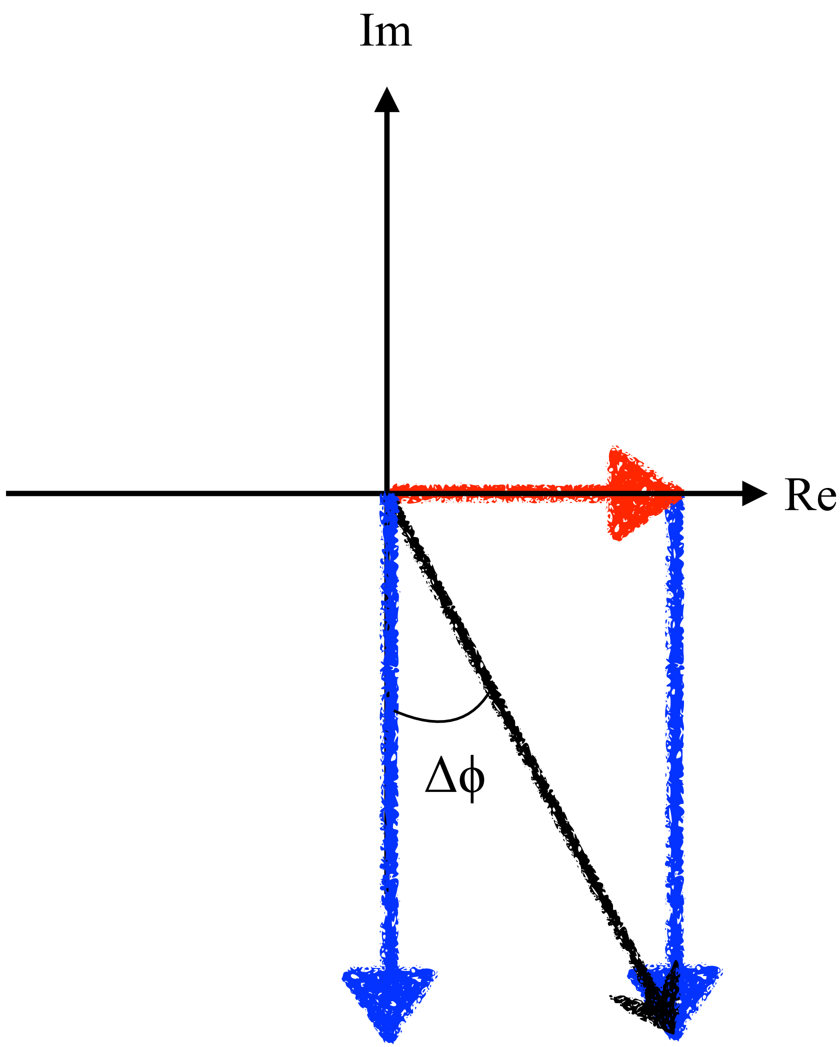

Increasing ω makes $|X_C| = 1/\omega C$ smaller. $V_C \propto 1/\omega$.

Larger voltage drop across C means V_{in} and V_{out} closer together in angle, smaller phase difference.

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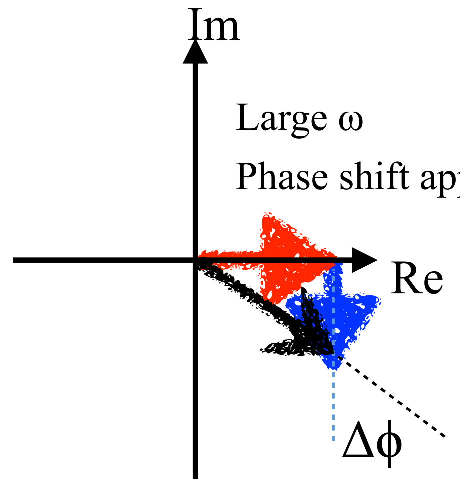
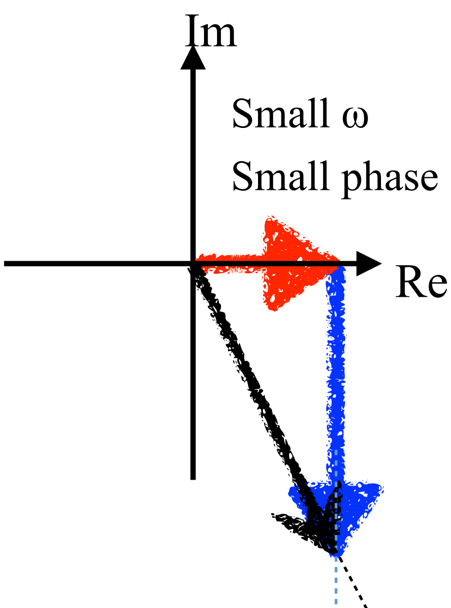
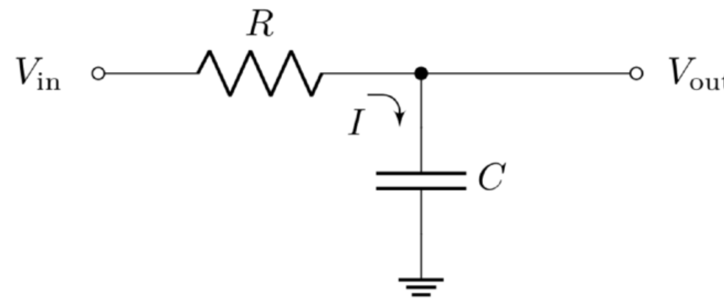
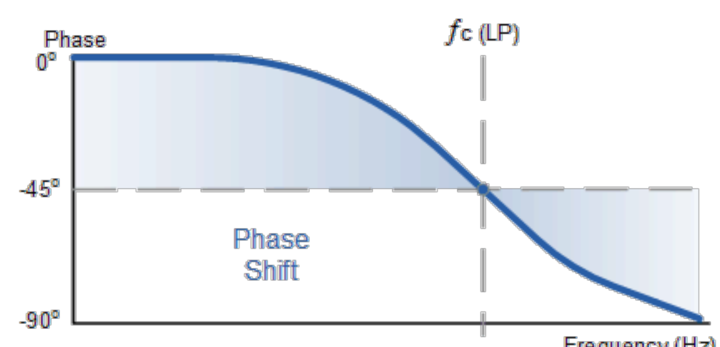






Small voltage drop across C has larger angle between the C vector, which is V_{out}, and the total vector, which is V_{in}.

Larger voltage drop across C means Vin and Vout closer together in angle, smaller phase difference.



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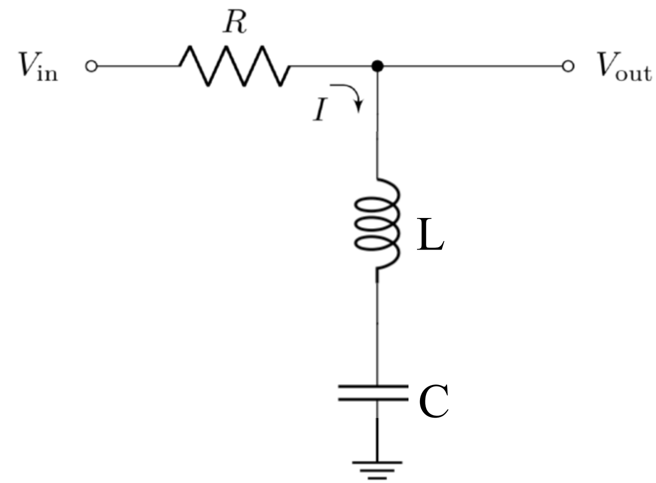

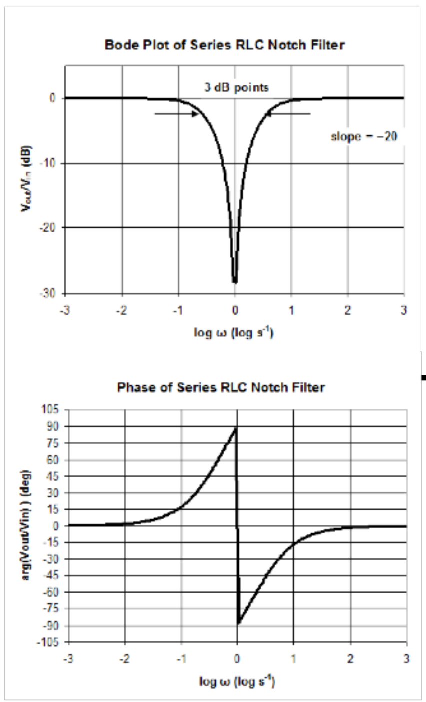

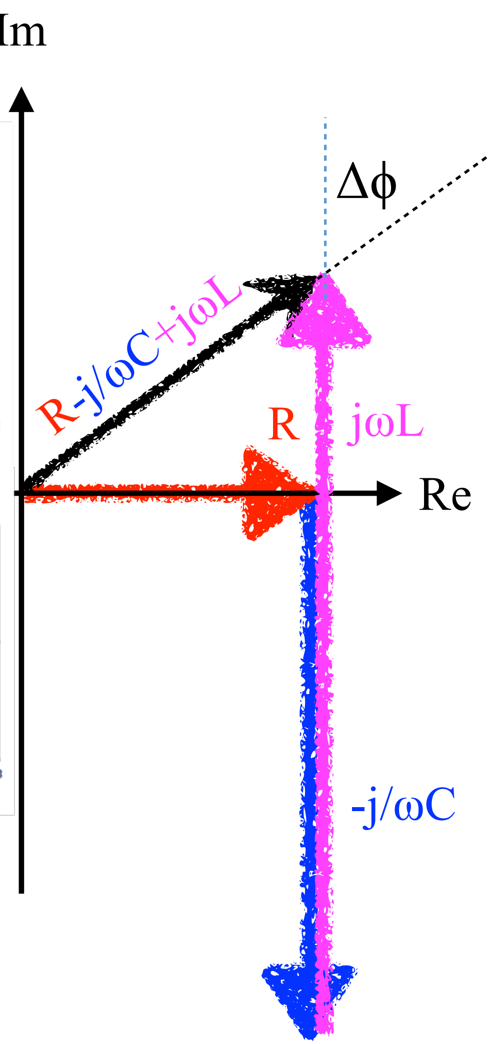

We can analyze an RLC circuit by combining the X_L and X_C.

When X_L and X_C approach exact cancellation, $V_{out} \rightarrow 0$ and phase difference $\rightarrow$ 90 degrees. When they deviate in either direction, the phase difference is positive or negative.

$$\omega L - 1/\omega C \rightarrow 0$$